A non-linear elastic constitutive framework for replicating plastic deformation in solids

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A non-linear elastic constitutive framework for replicating plastic deformation in solids

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Abstract

Ductile metals and other materials typically deform plastically under large applied loads; a behavior most often modeled using plastic deformation constitutive models. However, it is possible to capture some of the key behaviors of plastic deformation using only the framework for non-linear elastic mechanics. In this paper, we develop a phenomenological, hysteretic, nonlinear elastic constitutive model that captures many of the features expected of a plastic deformation model. This model is based on calculating a secant modulus directly from a material’s stress-strain curve. Scalar stress and strain values are obtained in three dimensions by using the von Mises invariants. Hysteresis is incorporated by tracking an additional history variable and assuming an elastic unloading response. This model is demonstrated in both single- and multi-element simulations under varying strain conditions.
# Contents

1 Introduction 7

2 Model 9
   2.1 Governing equations ................................................................. 9
   2.2 Model development ................................................................. 9
   2.3 Numerical implementation ....................................................... 11

3 Numerical examples 13
   3.1 Single element test .............................................................. 13
   3.2 Tensile test ............................................................................ 14
   3.3 Repeated stress-unstress cycles ............................................. 15

4 Conclusions 19
List of Figures

2.1 A typical stress-strain curve for a ductile metal undergoing a stress loading followed by an unloading. The true behavior is shown in black, with the dashed curve showing the response when stress is being released. The red curves show the path taken by this present model, which is achieved through use of the secant modulus, $E_s$, shown by the blue dashed-dot line. Green numbered points are discussed in the text. ................................................................. 10

3.1 True stress and true strain from the single-element test of §3.1 showing an applied tension followed by return to the undeformed state, compared to the input data [16]. 14

3.2 Plot of the secant modulus introduced in §2.2 that was used in the code for the single-element simulation of §3.1. ................................................................. 15

3.3 Simulation images of a aluminum cylinder before (left) and after (right) the tensile test described in §3.2. Necking behavior is observed in the final image. The objects are colored by the true von Mises strain. ................................................................. 16

3.4 True stress and strain as a function of the time and applied displacement for the tensile test. Strain values are shown on the left axis in black, while stress values are on the right axis in red. ................................................................. 17

3.5 Resulting stress-strain behavior from three repeated loading-unloading cycles, as described in §3.3. Black symbols represent the first loading-unloading cycle, red symbols the second cycle, and blue symbols the third cycle. The data is from [16, Fig. 9]. ................................................................. 17
Chapter 1

Introduction

Many materials, particularly ductile metals such as aluminum, show two distinct regimes of deformation under an applied stress. At lower values of stress the material will deform elastically, which can be easily captured in a linear elastic model. This strain is reversible, with the material returning to the initial state (no deformation) when the stress is removed. When the applied stress surpasses a critical value known as the yield stress ($\sigma_y$), the material response drastically changes, exhibiting two key characteristics. First, much less stress is required to strain the material a given amount, or an apparent softening of the material. Secondly, this additional strain is permanent, meaning the material does not return to its original configuration upon removal of the applied stress. The exact nature of this behavior is often determined experimentally using a tensile test apparatus [1] and is expressed in a stress-strain curve, as shown in Figure 2.1. The key features of this curve will be discussed in §2.2. For high strain rate experiments, the Hopkinson bar apparatus is typically used [7, 11].

Many models exist for capturing elastoplastic material response [10, 12]. These models are typically derived from continuum thermodynamic theories [6]. With this rigor, however, comes additional complexity in the form of tensor-based yield conditions, flow rules, and consistency conditions. This complexity also causes these models to be computationally expensive. Because of these issues, rigorous elastoplastic models are typically found in codes specialized for solid mechanics (e.g. ANSYS Solid Mechanics [2]) rather than in general finite element codes (e.g. GOMA [15]) that can be used for coupled multi-physics simulations. However, these more general codes often include simple linear or nonlinear elastic deformation models, often with small-strain formulations [13].

For these reasons, there is an appeal to mimicking plastic-like deformation within the framework of an elastic model. There is a long history of using nonlinear elastic constitutive equations to model complex behaviors in solids [19]. A pertinent example of this is known as secant elasticity or pseudo-elasticity, where the nonlinear constitutive equation is a fourth-order secant stiffness tensor, $\sigma = E_{\sim} : \varepsilon$. In this equation, $\sigma$ is the stress tensor, $\varepsilon$ is the strain tensor, and $E_{\sim}$ is the secant stiffness tensor. However, these models do not typically account for the hysteresis seen in plastic deformation. Alternatively, hypoelastic or tangential stiffness models represent the constitutive model in a differential form, $\dot{\sigma} = E_{\sim} : \dot{\varepsilon}$, where $E_{\sim}$ is the tangential stiffness tensor. These models do easily include path-dependencies, but the models themselves can be quite complex. While these nonlinear elastic models are often more simple than elastoplastic models, fourth-order tensors and hypoelastic response functions are still quite complex. We seek a simpler approach that uses the
standard hyperelastic formulation to model hysteretic plastic deformation.

A related, but inverse, approach to using nonlinear elastic constitutive equations to model plastic deformation is the field of using yield-stress fluid models or pseudoplastic models within an Eulerian fluid dynamics simulation [4, 14]. While many specific models exist, two of the most popular are the Herschel-Bulkley [9] and Bingham [3] models. In these models, unyielded regions (which are solid-like) are represented by a fluid with a very high viscosity. While the entire material region remains “fluid”, governed by the Navier-Stokes equations, the high viscosity requires a very high applied stress to cause significant flow, rendering it effectively immobile. The inverse problem that we address here is to treat the entire material region as “solid”, but for plastic flow regimes, the effective modulus is low enough to allow significant deformation.

In this report, we describe a simple new nonlinear elastic constitutive model for mimicking the hysteretic, plastic-like deformation observed in ductile materials, particularly those under tension. This model is directly calibrated from a material’s experimental stress-strain behavior under standard tests. In §2.1, we describe the basic equations of motion for solid deformation and elastic constitutive models. Our nonlinear constitutive model for plastic-like behavior is described in §2.2, and its numerical implementation is presented in §2.3. Finally, this model is tested using three numerical examples in §3.
Chapter 2

Model

2.1 Governing equations

Movement of the material is governed by the Cauchy momentum conservation equation,

\[ \rho \ddot{u} = \nabla \cdot \sigma + F, \]  

(2.1)

where \( \rho \) is the material density, \( u = x - X \) is the material displacement, \( x \) is the current material coordinates, \( X \) is the original material coordinates, \( \ddot{u} \) is the second time derivative of \( u \), \( \sigma \) is the Cauchy total stress tensor, and \( F \) is any applied body force. A linear elastic constitutive equation is used for the stress tensor,

\[ \sigma = \lambda \varepsilon \mathbb{I} + 2\mu \varepsilon, \]  

(2.2)

where \( \lambda \) and \( \mu \) are the first and second Lamé parameters, \( \varepsilon \) is the strain tensor, \( \varepsilon = \text{tr}(\varepsilon) \) is the volume change of the material, \( \text{tr}(\cdots) \) represents the trace of a matrix, and \( \mathbb{I} \) is the identity tensor. Small-strain theory is used to calculate the strain tensor,

\[ \varepsilon = \left( \nabla u + (\nabla u)^T \right). \]  

(2.3)

They key difference between the standard linear elastic model and the present model is that in the present model, the Lamé parameters (\( \lambda, \mu \)) are not constant. In §2.2, we describe the development of a model for \( \lambda \) and \( \mu \) that is based on the Young’s modulus, \( E \). The Lamé parameters can be directly calculated from \( E \) and the Poisson’s ratio, \( \nu \), using the following expressions

\[ \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \]  

(2.4)

\[ \mu = \frac{E}{2(1+\nu)} \]  

(2.5)

2.2 Model development

In order to get plastic-like behavior from an elastic model, the elastic modulus must dynamically change as a nonlinear function of the material strain. The idealized behavior for plastic deformation
Figure 2.1: A typical stress-strain curve for a ductile metal undergoing a stress loading followed by an unloading. The true behavior is shown in black, with the dashed curve showing the response when stress is being released. The red curves show the path taken by this present model, which is achieved through use of the secant modulus, $E_s$, shown by the blue dashed-dot line. Green numbered points are discussed in the text.

is frequently expressed in a stress-strain curve, as shown in Figure 2.1 by the solid black curve. At low values of strain the material behaves elastically with Young’s modulus $E$, as seen in the straight part of the curve. Eventually, the slope of the curve begins to decrease, a point known as the yield stress. This begins the plastic deformation regime, where the stress increases much more slowly with strain. In the elastic regime, if the stress is released, the material fully responds to zero strain. However in the plastic regime, some of the deformation/strain is unrecoverable, and the material typically responds elastically along the black dashed line, typically with an elastic modulus similar to the original modulus, $E$.

Typical elastic models cannot replicate this nonlinear, hysteretic relationship. In order to accommodate plastic-like behavior, we introduce the secant modulus, $E_s$, shown by the blue line in Figure 2.1. The secant modulus is simply the ratio of the stress from the stress-strain curve to the strain needed to achieve that stress, $E_s = \frac{\sigma}{\varepsilon}$ [5]. If this secant modulus is used as the Young’s modulus to calculate the Lamé parameters $\lambda$ and $\mu$ for the linear elastic constitutive equation (2.2), then the stress provided by the constitutive equation for a given input strain is exactly what is predicted by the stress-strain curve, thereby replicating the plastic behavior from the stress-strain curve. In Figure 2.1, the green annotation #1 is shown in the elastic regime, where the secant modulus is identical to the original elastic modulus. However, annotation #2 is in the plastic regime, and the secant modulus is lower than the original modulus, yielding more strain for a given level
of stress than the original elastic modulus. This model follows the red curve in Figure 2.1.

This procedure works as described for as long as the applied stress is constant or increasing, i.e. during the plastic deformation loading phase. In order to achieve a consistent elastic response when stress is relieved (annotation #4 in Figure 2.1), additional calculations must be made. First, the maximum strain over the history of the simulation ($\varepsilon_{\text{max}}$, annotation #3) must be tracked. Then, the current strain is compared to the maximum strain, and if the current strain is less than the maximum strain, then we assume the “downslope” behavior, rather than the “upslope” behavior. In this case, we want the material to follow the dashed red curve in Figure 2.1, which is again accomplished by using $E_s$, but this time calculated from the dashed curve, which has a slope of $E$. In this circumstance the secant modulus is

$$E_s = \frac{E(\varepsilon - \varepsilon_{\text{max}}) + \sigma_{\text{max}}}{\varepsilon_{\text{max}}}.$$ (2.6)

One key restriction to working within the framework of an elastic constitutive model is that the strain must return to zero if the stress is completely removed, i.e. the material must return to the origin of Figure 2.1. This is accomplished through the definition of a minimum modulus, $E_{\text{min}}$. If the calculated value of $E_s$ is less than $E_{\text{min}}$, then $E_{\text{min}}$ is used instead. This will require that at very low stresses, the material will respond as in annotation #5. While this behavior is not how a true plastic material responds, it is a necessity of the linear elastic constitutive model. Ideally, $E_{\text{min}}$ would be minimized to push this aphysical behavior to even lower stress values. However, this will make the equation system much more nonlinear, reducing numerical convergence.

To this point, this model has been laid out in terms of scalar stress and strain quantities, $\sigma$ and $\varepsilon$. However, in a computational simulation, the state of stress and strain are represented by tensors, $\sigma$ and $\varepsilon$. In order to relate the tensor quantities to the necessary scalar quantities, a tensor invariant must be used. Many stress tensor invariants are discussed in the literature, but the most common invariant for yield stress materials is the von Mises invariant [18],

$$\sigma_{\text{vm}} = \sqrt{\frac{3}{2} \sigma_{\text{dev}} : \sigma_{\text{dev}}}$$ (2.7)

This invariant is related to the second-invariant of the deviatoric part of the stress tensor, where $\sigma_{\text{dev}} = \sigma - \text{tr}(\sigma)I$ is the deviatoric part of the stress tensor. The corollary for this for the strain tensor invariant will also be called the von Mises strain,

$$\varepsilon_{\text{vm}} = \sqrt{\frac{3}{2} \varepsilon_{\text{dev}} : \varepsilon_{\text{dev}}}$$ (2.8)

where $\varepsilon_{\text{dev}} = \varepsilon - \text{tr}(\varepsilon)I$.

### 2.3 Numerical implementation

This model is implemented in the Galerkin/Finite Element Method code GOMA [15]. The equations are solved implicitly using fully-coupled nonlinear Newton iterations on 3-D on hexahedral
trilinear (Q1) elements. Linear systems are solved in serial using UMF, implemented via the Trilinos solver library [8].

In order to calculate and store the maximum time-history of strain ($\varepsilon_{\text{max}}$) necessary for (2.6), both $\varepsilon_{\text{max}}$ and $\varepsilon_{\text{vm}}$ are projected to the nodes and solved as part of the matrix system. While the maximum strain could be directly calculated and stored at each gauss point, this projection method ensures that the field remains smooth and simplifies the data structures necessary for storing the information.

The following simplified algorithm for calculating $E_s$ is performed at each material point for every time step:

1: Calculate $\varepsilon_{\text{vm}}$ and $\varepsilon_{\text{max}}$
2: if $\varepsilon_{\text{vm}}$ $\geq$ $\varepsilon_{\text{max}}$ then
3: Calculate $\sigma_{\text{vm}}$ from the stress-strain curve
4: Calculate $E_s = \sigma_{\text{vm}} / \varepsilon_{\text{vm}}$
5: else if $\varepsilon_{\text{vm}} < \varepsilon_{\text{max}}$ then
6: Calculate $E_s$ using (2.6)
7: end if
8: if $E_s < E_{\text{min}}$ then
9: $E_s = E_{\text{min}}$
10: end if
11: Calculate $\lambda$ and $\mu$ from $E_s$ and $\nu$ using (2.4) and (2.5)
Chapter 3

Numerical examples

In this section, we test this model using three example problems. First, we demonstrate the model on a single hexahedral element with a linear strain increase (loading) followed by a corresponding strain decrease (unloading) in §3.1. We then show a typical tensile test of a cylindrical rod in §3.2, testing the behavior at large strains by emulating a standard tensile test [1]. Finally, in §3.3, the behavior under repeated applications of a load-unload cycle is illustrated on a single element.

For all of these simulations, the stress-strain curve from [16, Fig. 9] for aluminum 6061-T6 in tension at a rate of 1000s$^{-1}$ is used. Numerical data (true stress and true strain) was extracted from [16, Fig. 9] using DataThief [17]. Simulations were run using GOMA [15] using the CGS unit system. The material was assumed to have $\nu = 1/3$ and $E_{\text{min}} = 10^8$ Barye. The yield stress extracted from the data is $\sigma_y = 2.6 \times 10^9$ Barye.

3.1 Single element test

Initial demonstration of this model uses single trilinear hexahedral element of size $1 \text{cm}^3$. The $y = 0 \text{cm}$ surface is held fixed by setting the displacement to zero. The $y = 1 \text{cm}$ surface is subjected to a linear displacement velocity of $0.03 \text{cm/s}$ for $1 \text{s}$, leading to a maximum displacement of $0.03 \text{cm}$ (tensile loading). At $t = 1 \text{s}$, the displacement function is then reversed, with a velocity of $-0.03 \text{cm/s}$ for another $1 \text{s}$, returning the displacement to zero (unloading).

The results of this test, in terms of the maximum von Mises stress as a function of the applied strain, are shown in Figure 3.1. The loading phase ($t = 0 - 1 \text{s}$) is shown with + symbols, the unloading phase ($t = 1 - 2 \text{s}$) is shown with o symbols, and the data is shown as the solid curve.

During the first part of the loading phase (when the stress is below the yield stress $\sigma_y$), the response is purely elastic with a modulus of $E = 5.7 \times 10^{11}$ Barye, as seen by the linear relationship between stress and strain in Figure 3.1. Once the yield stress is reached, however, the material begins responding in a nonlinear fashion, exactly following the experimental data. This is the plastic-like deformation that was the target of this model.

Once the unloading process begins, the material again responds elastically, with the same modulus as is observed during the loading phase. This response is intended and physical. However, once the minimum secant modulus ($E_{\text{min}}$) is reached, the model fails to follow the expected physi-
Figure 3.1: True stress and true strain from the single-element test of §3.1 showing an applied tension followed by return to the undeformed state, compared to the input data [16].

cal response, namely that there is a permanent deformation under zero applied stress. Instead, the strain quickly returns to zero, the primary deficiency with this model.

The reason for this aphysical behavior can be clearly seen in Figure 3.2, which shows the effective secant modulus as a function of the strain. While the secant modulus remains reasonable during the loading phase, as the stress approaches zero during the unloading phase, the secant modulus begins to shrink exponentially. This necessitates a clipping value, $E_{\text{min}}$. The value of $E_{\text{min}}$ should be chosen low enough to not significantly affect the desired behavior while still retaining computational convergence.

### 3.2 Tensile test

In this example, we reproduce a standard tensile test [1] that is often used to generate stress-strain curves for materials. In this test, a cylindrical sample, shown in Figure 3.3, is grabbed at both ends and then deformed along the primary (y) axis. For this simulation, the cylindrical sample is of 3 cm length and 0.5 cm radius and the top surface is deformed at a rate of 0.05 cm/s for 6 s.

A characteristic behavior of ductile metals in these tests is necking, where the mid-section of the sample thins at a much higher rate than the ends. This necking behavior is also seen in our simulation, shown in the second image of Figure 3.3. This figure is colored by the von Mises strain, which you can see is greatest in the center of the sample.
This plastic deformation and necking behavior is quantified in Figure 3.4, which shows the maximum von Mises stress and strain as a function of time. Many regimes of behavior are observed here. For the first few time steps, the sample deforms elastically. Once the yield stress is reached, however, plastic-like deformation is observed, shown by the turnover in the stress curve at \( \sim 0.5 \text{s} \), a similar behavior to that seen in §3.1. At approximately 3s, the slope of the strain curve increases. This indicates the onset of necking, where the sample begins to deform nonuniformly. As Figure 3.4 is showing the maximum values of stress and strain, this change of slope indicates that more of the applied deformation is being taken up in a given region (the center) than in other parts, since the overall deformation rate is constant. Finally, at approximately 4s, the maximum stress decreases then levels off. Now, nearly all of the deformation is observed in the necked region and failure is probably imminent.

### 3.3 Repeated stress-unstress cycles

For this final example, we use the same geometry that was used in §3.1, a single hexahedral element. In this example, however, we subject the sample to repeated loading-unloading cycles, each with an increasing maximum deformation. This is a common experiment to probe how plastic strain may affect the elastic response of a material. In the current model, however, we do not include strain-softening or -hardening. Still, this remains a good example simulation to test how our model will capture the hysteresis that’s expected from this situation.
Figure 3.3: Simulation images of a aluminum cylinder before (left) and after (right) the tensile test described in §3.2. Necking behavior is observed in the final image. The objects are colored by the true von Mises strain.

In this example, top of the element is deformed 0.02 cm, then the deformation is removed back to zero, a cycle very similar to that of §3.1, and shown with the black symbols in Figure 3.5. Then, the element is deformed twice as much, to 0.04 cm, then the deformation again removed to zero, shown with the red symbols in Figure 3.5. Finally, the element is deformed to three times the initial deformation, to 0.06 cm, then removed, as shown with the blue symbols in Figure 3.5.

The first cycle behaves identically to the problem described in §3.1, with an initial elastic followed by plastic deformation, then an elastic response during unloading, followed by the aphysical return to zero deformation once the minimum modulus $E_{\text{min}}$ is reached. On the second cycle, however, a much different behavior is observed. Since the model has a memory of the maximum strain, and therefore has hysteresis, the loading part of the second cycle initial follows the unloading curve of the first cycle, with an aphysical response at first, followed by an elastic deformation until the previous maximum strain is reached. Once this maximum strain is reached, the material begins plastically deforming again, following the data-based stress-strain curve. On unloading, the response is similar to the first cycle, with an elastic response until $E_{\text{min}}$ is reached. The third cycle shows identical behaviors to the second cycle.
Figure 3.4: True stress and strain as a function of the time and applied displacement for the tensile test. Strain values are shown on the left axis in black, while stress values are on the right axis in red.

Figure 3.5: Resulting stress-strain behavior from three repeated loading-unloading cycles, as described in §3.3. Black symbols represent the first loading-unloading cycle, red symbols the second cycle, and blue symbols the third cycle. The data is from [16, Fig. 9].
Chapter 4

Conclusions

In this paper, we presented a novel constitutive model for representing plastic behavior within the constraints of an elastic simulation code. This model is based upon calculating the secant modulus to any point on the material’s stress-strain curve and using that as the elastic modulus. The von Mises invariants are used to calculate the scalar stress and strain values used for the elastic modulus.

While we recognize that numerous rigorous theories and models of plastic deformation exist and are readily available, they were not implemented in GOMA, while the elastic deformation framework was heavily tested and robust. This novel model provides a quick and computationally-efficient method to reproduce a limited set of plastic behaviors without the need to implement a full plastic deformation equation set. It was developed to mimic ductile metal behavior well before the maximum strength of the material. We recognize that the current implementation uses small-strain theory, which is not objective. However, the model can be calibrated to experimental data and gives reasonable results, considering the constraints.

Three example problems were presented to demonstrate the abilities of this model. First, a single element simulation was used to demonstrate that plastic-like behaviors are observed and the model can be used to follow a material’s stress-strain curve into the plastic regime. Then, larger deformations were demonstrated using the standard tensile-test geometry. Finally, the single element was subjected to repeated stressings, showing that the model can handle the desired anisotropy of behavior.
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