A Set of Verification Test Cases for Eiger: Plane Wave Scattering from a Sphere

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Abstract

This report discusses a set of verification test cases for the frequency-domain, boundary-element, electromagnetics code Eiger based on the analytical solution of plane wave scattering from a sphere. Three cases will be considered: when the sphere is made of perfect electric conductor, when the sphere is made of lossless dielectric and when the sphere is made of lossy dielectric. We outline the procedures that must be followed in order to carefully compare the numerical solution to the analytical solution. We define an error criterion and demonstrate convergence behavior for both the analytical and numerical cases. These problems test the code’s ability to calculate the surface current density and secondary quantities, such as near fields and far fields.
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A Set of Verification Test Cases for Eiger:
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1 Introduction

In this report we compare solutions obtained from the frequency-domain, boundary-element, electromagnetics code Eiger to analytical solutions of plane wave scattering from a sphere. Three types of spheres will be considered: a perfect electric conductor (PEC), lossless dielectric and lossy dielectric.

We begin by defining the general test case geometry. We then derive the analytical solution of plane wave scattering from a PEC sphere and calculate all components of both $E$ and $H$ fields in spherical coordinates at locations outside the sphere. By allowing the observation point to be on the surface of the sphere, we obtain both components of electrical current density, which are the principal unknowns calculated by Eiger. By allowing the radius of the observation point to approach infinity, we obtain the $E$ field components in the far field. In Section 3.4, we perform a convergence study of the analytical solution for all quantities calculated. The procedures used to compare the numerical solution to the analytical solution are quite involved and are, therefore, documented in Section 3.5. Comparisons between numerical and analytical solutions are made for all quantities at various frequencies to show how the error varies as a function number of basis functions per wavelength. Results are shown for the three equations that are implemented in Eiger to solve the PEC boundary condition: the electric field integral equation (EFIE), the magnetic field integral equation (MFIE) and the combined field integral equation (CFIE). We also demonstrate how the CFIE overcomes problems that the EFIE and MFIE have at internal resonance. Comparing results of Eiger to the analytical calculation of currents on the PEC sphere is one of the verification tests required for the electromagnetic radiation (EMR) environment (Test 1.a.a) [1].

The analytic solution to plane wave scattering from a dielectric sphere is discussed in Section 4. Convergence studies, comparison procedures and comparisons to numerical solutions follow the same format as the PEC sphere. The lossy dielectric sphere discussion starts in Section 5 and follows the same format as the other two test cases.

2 Test Case Geometry

Consider a plane wave travelling in the $+\hat{\mathbf{z}}$ direction incident on a sphere as shown in Figure 1. The $\mathbf{E}$ field is polarized along $\hat{x}$ so

$$\mathbf{E} = \hat{x}E_0e^{-jk_0z}$$

where $k_0 = \omega\sqrt{\varepsilon_0\mu_0}$ is the wavenumber in free-space (throughout this report, we assume an $e^{+j\omega t}$ time dependency). The sphere is centered at the origin and has a radius $a$. 
3 PEC Sphere Test Case

3.1 Derivation of the Analytical Solution

If the sphere is a PEC, the potentials at \( r \geq a \) are given as [2]

\[
A_r^+ = \frac{E_0}{\omega \mu_0} \cos \phi \sum_{n=1}^{\infty} \left[ a_n \tilde{J}_n(k_0 r) + b_n \tilde{H}_n^{(2)}(k_0 r) \right] P_n^1(\cos \theta)
\]

\[
F_r^+ = \frac{E_0}{k_0} \sin \phi \sum_{n=1}^{\infty} \left[ a_n \tilde{J}_n(k_0 r) + c_n \tilde{H}_n^{(2)}(k_0 r) \right] P_n^1(\cos \theta)
\]

where \( r, \theta \) and \( \phi \) are the spherical coordinates; \( P_n^1 \) is the associated Legendre polynomial of order 1 and degree \( n \); \( \tilde{J}_n \) is the alternative spherical Bessel function of the first kind, order \( n \); \( \tilde{H}_n \) is the alternative spherical Hankel function of the second kind, order \( n \); and \( \mu_0 \) is the permeability of the medium surrounding the sphere. Calculation of \( P_n^1, \tilde{J}_n \) and \( \tilde{H}_n \), along with their derivatives, are discussed in Appendix I.
Additionally,
\[ a_n = \frac{j^{-n}(2n+1)}{n(n+1)} \]
\[ b_n = -a_n \frac{\tilde{J}_n(k_0a)}{H_n^{(2)}(k_0a)} \]
\[ c_n = -a_n \frac{\tilde{J}_n(k_0a)}{H_n^{(2)}(k_0a)} \]

The total field components are calculated from these potentials. For the \(E\) field components
\[
E_r = \frac{1}{jw\varepsilon_0} \left( \frac{\partial^2}{\partial r^2} + k_0^2 \right) A_r^+ = -jE_0 \cos \phi \sum_{n=1}^{\infty} \left[ a_n \left( \frac{\partial^2}{\partial r^2} + k_0^2 \right) \tilde{J}_n(k_0r) + b_n \left( \frac{\partial^2}{\partial r^2} + k_0^2 \right) \tilde{H}_n^{(2)}(k_0r) \right] P_n^1(\cos \theta)
\]
\[ E_\theta = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left( \sin \phi \right) \sum_{n=1}^{\infty} \left[ a_n \tilde{J}_n(k_0r) + b_n \tilde{H}_n^{(2)}(k_0r) \right] P_n^1(\cos \theta)
\]
\[ E_\phi = \frac{-jE_0}{r \sin \theta} \frac{\partial}{\partial \phi} \left( \sin \phi \right) \sum_{n=1}^{\infty} \left[ a_n \tilde{J}_n(k_0r) + b_n \tilde{H}_n^{(2)}(k_0r) \right] P_n^1(\cos \theta) \sin \theta
\]

At \(r = a\), and substituting for \(b_n\) and \(c_n\) in terms of \(a_n\) we obtain
\[
E_\theta = \frac{-E_0 \cos \phi}{k_0a} \sum_{n=1}^{\infty} \left[ a_n \tilde{J}_n(k_0a) - a_n \frac{\tilde{J}_n(k_0a)}{H_n^{(2)}(k_0a)} \tilde{H}_n^{(2)}(k_0a) \right] \frac{P_n^1(\cos \theta)}{\sin \theta}
\]
\[ -j \left[ a_n \tilde{J}_n(k_0a) - a_n \frac{\tilde{J}_n(k_0a)}{H_n^{(2)}(k_0a)} \tilde{H}_n^{(2)}(k_0a) \right] P_n^1(\cos \theta) \sin \theta
\]
\[ = \frac{-E_0 \cos \phi}{k_0a} \sum_{n=1}^{\infty} \left[ \sin \theta \right] P_n^1(\cos \theta) - j \left[ \sin \theta \right] P_n^1(\cos \theta) \sin \theta
\]
\[ = 0
\]

which states that the PEC boundary condition (\(E\) tangential to a PEC surface is zero) is satisfied. The scattered \(E_\theta\) field is
\[
E_\theta^s = \frac{-E_0 \cos \phi}{k_0r} \sum_{n=1}^{\infty} \left[ c_n j^{-n+1} e^{-jk_0r} \right] \frac{P_n^1(\cos \theta)}{\sin \theta} - j \left[ b_n \tilde{H}_n^{(2)'}(k_0r) \right] P_n^1(\cos \theta) \sin \theta
\]
which, when \(kr \to \infty\), becomes
\[
E_\theta^s = \frac{jE_0}{k_0r} e^{-jk_0r} \cos \phi \sum_{n=1}^{\infty} \left( b_n P_n^1(\cos \theta) \sin \theta - c_n \frac{P_n^1(\cos \theta)}{\sin \theta} \right)
\]

13
The total $E_\phi$ field is

$$E_\phi = \frac{1}{r} \frac{\partial F_r^{(1)}}{\partial \varphi} + \frac{1}{\sqrt{-\varepsilon_0 \mu_0}} \frac{\partial^2 A_r^{(1)}}{\partial r \partial \phi}$$

$$= \frac{E_0 \sin \phi}{r k_0} \sum_{n=1}^\infty \left[ a_n \hat{J}_n(k_0r) + c_n \hat{H}_n^{(2)}(k_0r) \right] \frac{\partial}{\partial \theta} \left( P_n^1(\cos \theta) \right)$$

$$- \frac{j E_0}{k_0 r \sin \theta} \sum_{n=1}^\infty \left[ a_n \hat{J}_n(k_0r) + b_n \hat{H}_n^{(2)}(k_0r) \right] P_n^1(\cos \theta)$$

$$= -\frac{E_0 \sin \phi}{k_0 r} \sum_{n=1}^\infty \left[ a_n \hat{J}_n(k_0r) + c_n \hat{H}_n^{(2)}(k_0r) \right] P_n^1(\cos \theta) \sin \theta - j \left[ a_n \hat{J}_n(k_0r) + b_n \hat{H}_n^{(2)}(k_0r) \right] \frac{P_n^1(\cos \theta)}{\sin \theta}$$

At $r = a$, the PEC boundary condition states that total $E_\phi = 0$, i.e.,

$$E_\phi = -\frac{E_0 \sin \phi}{k_0 a} \sum_{n=1}^\infty \left[ a_n \hat{J}_n(a) - a_n \hat{J}_n(a) \hat{H}_n^{(2)}(a) \right] P_n^1(\cos \theta) \sin \theta$$

$$- j \left[ a_n \hat{J}_n(a) - a_n \hat{J}_n(a) \hat{H}_n^{(2)}(a) \right] \frac{P_n^1(\cos \theta)}{\sin \theta}$$

$$= 0$$

The scattered $E_\phi$ field is

$$E_\phi^s = -\frac{E_0 \sin \phi}{k_0 r} \sum_{n=1}^\infty \left[ c_n \hat{H}_n^{(2)}(k_0r) \right] P_n^1(\cos \theta) \sin \theta - j \left[ b_n \hat{H}_n^{(2)}(k_0r) \right] \frac{P_n^1(\cos \theta)}{\sin \theta}$$

which, when $kr \to \infty$, becomes

$$E_\phi^s = -\frac{E_0 \sin \phi}{k_0 r} \sum_{n=1}^\infty \left[ c_n j^{n+1} e^{-j k_0 r} \right] P_n^1(\cos \theta) \sin \theta - j \left[ b_n j^{n+1} (-j) e^{-j k_0 r} \right] \frac{P_n^1(\cos \theta)}{\sin \theta}$$

$$= \frac{j E_0}{k_0 r} e^{-j k_0 r} \sin \phi \sum_{n=1}^\infty j^n \left[ b_n \frac{P_n^1(\cos \theta)}{\sin \theta} - c_n P_n^1(\cos \theta) \sin \theta \right]$$

The $H$ field components are

$$H_r = \frac{1}{j \omega \mu_0} \left( \frac{\partial^2}{\partial r^2} + k_0^2 \right) F_r^+$$

$$= -\frac{j E_0}{\sqrt{-\varepsilon_0 \mu_0} \sin \phi} \sum_{n=1}^\infty \left[ a_n \left( \frac{\partial^2}{\partial r^2} + k_0^2 \right) \hat{J}_n(k_0r) + c_n \left( \frac{\partial^2}{\partial r^2} + k_0^2 \right) \hat{H}_n^{(2)}(k_0r) \right] P_n^1(\cos \theta)$$

$$= -\frac{j E_0}{\eta_0} \sin \phi \sum_{n=1}^\infty \left[ a_n \hat{J}_n''(k_0r) + c_n \hat{H}_n^{(2)''}(k_0r) \right] P_n^1(\cos \theta) + \left[ a_n \hat{J}_n(k_0r) + c_n \hat{H}_n^{(2)}(k_0r) \right] P_n^1(\cos \theta)$$
\[ H_\theta = \frac{1}{r \sin \theta} \frac{\partial A_r^+}{\partial \phi} + \frac{1}{j \omega \mu_0 r} \frac{\partial^2 E_r^+}{\partial r \partial \theta} \]

\[ = \frac{E_0}{\omega \mu_0 r \sin \theta} \left[ \frac{\partial}{\partial \phi} (\cos \phi) \sum_{n=1}^{\infty} \left[ a_n \mathcal{J}_n (k_0 r) + b_n \mathcal{K}_n (k_0 r) \right] \frac{P_n^1 (\cos \theta)}{\sin \theta} \right] \]

\[ + \frac{E_0}{j \omega \mu_0 r \kappa_0} \left[ \sin \phi \sum_{n=1}^{\infty} \frac{\partial}{\partial r} \left[ a_n \mathcal{J}_n (k_0 r) + c_n \mathcal{K}_n (k_0 r) \right] \frac{\partial}{\partial \theta} \left( P_n^1 (\cos \theta) \right) \right] \]

\[ = -\frac{E_0 \sin \phi}{\eta_0 k_0 r} \sum_{n=1}^{\infty} \left[ a_n \mathcal{J}_n (k_0 r) + b_n \mathcal{K}_n (k_0 r) \right] \frac{P_n^1 (\cos \theta)}{\sin \theta} \]

\[ - j \left[ a_n \mathcal{J}_n (k_0 r) - a_n \frac{\mathcal{J}_n (k_0 a)}{\mathcal{K}_n (k_0 a)} \right] \frac{P_n^1 (\cos \theta)}{\sin \theta} \]

\[ = -\frac{E_0 \sin \phi}{\eta_0 k_0 a} \sum_{n=1}^{\infty} \left\{ \left[ \mathcal{J}_n (k_0 a) \mathcal{K}_n (k_0 a) - \frac{\mathcal{J}_n (k_0 a)}{\mathcal{K}_n (k_0 a)} \right] \frac{P_n^1 (\cos \theta)}{\sin \theta} \right\} \]

Simplifying the quantities in the brackets by invoking the Wronskian \( W = \mathcal{J} \mathcal{K} - \mathcal{K} \mathcal{J} = -j \) we obtain

\[ H_\theta (k_0 a) = -\frac{E_0 \sin \phi}{\eta_0 k_0 a} \sum_{n=1}^{\infty} \left\{ \left[ \frac{P_n^1 (\cos \theta)}{\mathcal{K}_n (k_0 a) \sin \theta} - \frac{P_n^1 (\cos \theta)}{\mathcal{K}_n (k_0 a) \sin \theta} \right] \right\} \]

\[ = \frac{j}{\eta_0} E_0 \sin \phi \sum_{n=1}^{\infty} \left\{ \frac{P_n^1 (\cos \theta)}{\mathcal{K}_n (k_0 a) \sin \theta} - \frac{P_n^1 (\cos \theta) \sin \theta}{\mathcal{K}_n (k_0 a) \sin \theta} \right\} \]

which allows us to calculate the \( \hat{\phi} \) directed electric current density:

\[ \hat{\phi} J_\phi = \hat{r} \times \hat{\theta} H_\theta \]

\[ J_\phi = \frac{j}{\eta_0} E_0 \sin \phi \sum_{n=1}^{\infty} \left\{ \frac{P_n^1 (\cos \theta)}{\mathcal{K}_n (k_0 a) \sin \theta} - \frac{P_n^1 (\cos \theta) \sin \theta}{\mathcal{K}_n (k_0 a) \sin \theta} \right\} \]

Equation 1 agrees with Equation (6-103) of [2].
\[ H_\phi = \frac{-1}{r} \frac{\partial A_r^+}{\partial \theta} + \frac{1}{j \omega \mu_0 r \sin \theta} \frac{\partial^2 F_r^+}{\partial r \partial \theta} \]

\[ = -\frac{E_0}{\omega \mu_0 r} \sum_{n=1}^{\infty} \left[ a_n \hat{J}_n(k_0 r) + b_n \hat{H}_{\phi}^{(2)}(k_0 r) \right] \frac{\partial}{\partial \theta} P_n^1(\cos \theta) \]

\[ + \frac{E_0}{j \omega \mu_0 k_0 r \sin \theta} \left( \sin \phi \right) \sum_{n=1}^{\infty} \left[ a_n \hat{J}_n(k_0 r) + c_n \hat{H}_{\phi}^{(2)}(k_0 r) \right] P_n^1(\cos \theta) \]

\[ = \frac{E_0 \cos \phi}{\eta_0 k_0 r} \sum_{n=1}^{\infty} \left[ a_n \hat{J}_n(k_0 r) + b_n \hat{H}_{\phi}^{(2)}(k_0 r) \right] P_n^{1'}(\cos \theta) \sin \theta - j \left[ a_n \hat{J}_n(k_0 r) + c_n \hat{H}_{\phi}^{(2)}(k_0 r) \right] \frac{P_n^1(\cos \theta)}{\sin \theta} \]

At \( r = a \),

\[ H_\phi = \frac{E_0 \cos \phi}{k_0 \eta_0 a} \sum_{n=1}^{\infty} \left[ a_n \hat{J}_n(k_0 a) - a_n \frac{\hat{J}_n(k_0 a)}{\hat{H}_{\phi}^{(2)}(k_0 a)} \hat{H}_n^{(2)}(k_0 a) \right] P_n^1(\cos \theta) \sin \theta \]

\[ - j \left[ a_n \hat{J}_n(k_0 a) - a_n \frac{\hat{J}_n(k_0 a)}{\hat{H}_{\phi}^{(2)}(k_0 a)} \hat{H}_n^{(2)}(k_0 a) \right] \frac{P_n^1(\cos \theta)}{\sin \theta} \]

\[ = \frac{E_0 \cos \phi}{k_0 \eta_0 a} \sum_{n=1}^{\infty} a_n \left\{ \left[ \hat{J}_n(k_0 a) \hat{H}_{\phi}^{(2)}(k_0 a) - \hat{J}_n(k_0 a) \hat{H}_n^{(2)}(k_0 a) \right] \frac{P_n^{1'}(\cos \theta) \sin \theta}{\hat{H}_{\phi}^{(2)}(k_0 a) \sin \theta} \right\} \]

\[ = -\frac{j}{\eta_0} \frac{E_0 \cos \phi}{k_0 a} \sum_{n=1}^{\infty} a_n \left\{ \frac{P_n^{1'}(\cos \theta) \sin \theta}{\hat{H}_{\phi}^{(2)}(k_0 a) \sin \theta} + \frac{j P_n^1(\cos \theta)}{\hat{H}_n^{(2)}(k_0 a) \sin \theta} \right\} \]

which allows us to calculate the \( \hat{\theta} \) directed electric current density:

\[ \hat{\theta} J_\theta = -\hat{r} \times \hat{\phi} \hat{H}_\phi \]

\[ J_\theta = \frac{j}{\eta_0} \frac{E_0 \cos \phi}{k_0 a} \sum_{n=1}^{\infty} a_n \left\{ \frac{P_n^{1'}(\cos \theta) \sin \theta}{\hat{H}_{\phi}^{(2)}(k_0 a) \sin \theta} + \frac{j P_n^1(\cos \theta)}{\hat{H}_n^{(2)}(k_0 a) \sin \theta} \right\} \]

Equation 2 agrees with Equation 6-103 in [2].

### 3.2 Summary of the Analytical Solution

#### 3.2.1 Currents

\[ J_\theta = \frac{j}{\eta_0} \frac{E_0 \cos \phi}{k_0 a} \sum_{n=1}^{\infty} a_n \left\{ \frac{P_n^{1'}(\cos \theta) \sin \theta}{\hat{H}_{\phi}^{(2)}(k_0 a) \sin \theta} + \frac{j P_n^1(\cos \theta)}{\hat{H}_n^{(2)}(k_0 a) \sin \theta} \right\} \]

\[ J_\phi = \frac{j}{\eta_0} \frac{E_0 \sin \phi}{k_0 a} \sum_{n=1}^{\infty} a_n \left\{ \frac{P_n^1(\cos \theta)}{\hat{H}_n^{(2)}(k_0 a) \sin \theta} - \frac{P_n^{1'}(\cos \theta) \sin \theta}{j \hat{H}_n^{(2)}(k_0 a) \sin \theta} \right\} \]

#### 3.2.2 Near Fields

\[ E_r = -j E_0 \cos \phi \sum_{n=1}^{\infty} \left[ a_n \hat{J}_n(k_0 r) + b_n \hat{H}_{\phi}^{(2)}(k_0 r) \right] P_n^1(\cos \theta) + \left[ a_n \hat{J}_n(k_0 r) + b_n \hat{H}_{\phi}^{(2)}(k_0 r) \right] P_n^1(\cos \theta) \]
$$E_\theta = \frac{-E_0 \cos \phi}{k_0 r} \sum_{n=1}^{\infty} \left[ a_n J_n(k_0 r) + c_n \tilde{H}_n(2)(k_0 r) \right] \frac{P_n^1(\cos \theta)}{\sin \theta} - j \left[ a_n J_n^\prime(k_0 r) + b_n \tilde{H}_n(2)(k_0 r) \right] \frac{P_n^{1\prime}(\cos \theta) \sin \theta}{\sin \theta}$$

$$E_\phi = \frac{-E_0 \sin \phi}{k_0 r} \sum_{n=1}^{\infty} \left[ a_n J_n(k_0 r) + c_n \tilde{H}_n(2)(k_0 r) \right] \frac{P_n^{1\prime}(\cos \theta) \sin \theta}{\sin \theta} - j \left[ a_n J_n^\prime(k_0 r) + b_n \tilde{H}_n(2)(k_0 r) \right] \frac{P_n^1(\cos \theta)}{\sin \theta}$$

$$H_r = \frac{-jE_0}{\eta_0} \sin \phi \sum_{n=1}^{\infty} \left[ a_n J_n(k_0 r) + c_n \tilde{H}_n(2)(k_0 r) \right] \frac{P_n^1(\cos \theta)}{\sin \theta} + \left[ a_n J_n^\prime(k_0 r) + c_n \tilde{H}_n(2)(k_0 r) \right] \frac{P_n^{1\prime}(\cos \theta) \sin \theta}{\sin \theta}$$

$$H_\theta = \frac{-E_0 \sin \phi}{\eta_0 k_0 r} \sum_{n=1}^{\infty} \left[ a_n J_n(k_0 r) + c_n \tilde{H}_n(2)(k_0 r) \right] \frac{P_n^1(\cos \theta) \sin \theta}{\sin \theta} - j \left[ a_n J_n^\prime(k_0 r) + c_n \tilde{H}_n(2)(k_0 r) \right] \frac{P_n^{1\prime}(\cos \theta)}{\sin \theta}$$

3.2.3 Far Fields

$$E^e_\theta = \frac{jE_0}{k_0 r} e^{-jk_0 r} \cos \phi \sum_{n=1}^{\infty} j^n \left\{ b_n P_n^1(\cos \theta) \sin \theta - c_n \frac{P_n^1(\cos \theta)}{\sin \theta} \right\}$$

$$E^e_\phi = \frac{jE_0}{k_0 r} e^{-jk_0 r} \sin \phi \sum_{n=1}^{\infty} j^n \left\{ b_n \frac{P_n^1(\cos \theta)}{\sin \theta} - c_n P_n^{1\prime}(\cos \theta) \sin \theta \right\}$$

3.3 Code Implementation of the Analytical Solution

The formulas in Section 3.2 were implemented in the code pec_sphere. Pec_sphere calculates quantities of interest versus \( \theta \), which varies from 1° to 179° in 1° increments. The other two observation coordinates, \( \phi_{\text{obs}} \) and \( r_{\text{obs}} \), are set by the user. In the code, \( E_0 = 1.0 \text{ V/m} \).

User input to pec_sphere is interactive and consists of the following:
- Number of terms \( N \) used in the summations.
- Frequency (Hertz).
- Sphere radius \( a \) (meters).
- Observation radius \( r_{\text{obs}} \) (meters).
- Observation phi \( \phi_{\text{obs}} \) (degrees).

The results are written to five output files, each of which contain 179 rows (1 row per \( \theta \) value) and seven columns. When necessary, zeros are put in certain columns of each row in order to make the format consistent. All angles (\( \theta \) and phase) are given in degrees.

Output files:
- pec_sphere_jt.txt: \( \theta \), \( |J_\theta| \), phase \( J_\theta \), 0.0, 0.0, 0.0, 0.0
- pec_sphere_jp.txt: \( \theta \), \( |J_\phi| \), phase \( J_\phi \), 0.0, 0.0, 0.0, 0.0
- pec_sphere_enf.txt: \( \theta \), \( |E_r| \), phase \( E_r \), \( |E_\theta| \), phase \( E_\theta \), \( |E_\phi| \), phase \( E_\phi \)
- pec_sphere_hnf.txt: \( \theta \), \( |H_r| \), phase \( H_r \), \( |H_\theta| \), phase \( H_\theta \), \( |H_\phi| \), phase \( H_\phi \)
- pec_sphere_ff.txt: \( \theta \), 0.0, 0.0, \( r e^{+jk_0 r} E^f_\theta \), phase \( r e^{+jk_0 r} E^f_\theta \), \( r e^{+jk_0 r} E^f_\phi \), phase \( r e^{+jk_0 r} E^f_\phi \).
The first two files (pec_sphere_jt.txt and pec_sphere_jp.txt) contain the \( \hat{\theta} \) and \( \hat{\phi} \) directed electric current density that exists on the surface of the sphere at \( \phi_{\text{obs}} \). The next two files (pec_sphere_enf.txt and pec_sphere_hnf.txt) contain the spherical components of total \( E \) and \( H \) fields in the near field region at \( (r_{\text{obs}}, \phi_{\text{obs}}) \). The last file (pec_sphere_ff.txt) contains the \( \hat{\theta} \) and \( \hat{\phi} \) directed components of \( E^a \) in the far field multiplied by the factor \( (r e^{+jk\rho}) \).

### 3.4 Convergence Study of the Analytical Solution

We will first examine the convergence behavior of the analytical solution as we increase the number of terms in the summations. Anticipating that this solution will eventually be compared to a numerical solution, we will use a sphere radius of \( a = 0.9989497 \) meters for reasons that will be explained later. We will observe all quantities at \( \phi_{\text{obs}} = 45^0 \). Near field quantities will be observed at \( r_{\text{obs}} = 1.1 \) meters. We will look at convergence for three different frequencies: \( f = 2.997925 \times 10^7 \) Hz \( (k_0a \approx 0.2\pi) \), \( f = 2.997925 \times 10^8 \) Hz \( (k_0a \approx 2.0\pi) \), and \( f = 2.997925 \times 10^9 \) Hz \( (k_0a \approx 20.0\pi) \), which span the frequency range of the numerical solutions.

In the following tables we compare the root mean squared relative error (\( \text{RE}_{\text{rms}} \)), which is defined as

\[
\text{RE}_{\text{rms}} = \sqrt{\frac{1}{179.3^0} \int_{0.5^0}^{179.3^0} \frac{|R_N(\theta) - R_{\text{exact}}(\theta)|^2}{|R_{\text{exact}}|^2} d\theta}
\]

where \( R_N(\theta) \) is the compared quantity \((J_\theta, E_\phi, \text{etc...})\) summed up to \( N \) modes, \( R_{\text{exact}}(\theta) \) is the exact quantity and \( R_{\text{exact}}^{\text{max}} \) is the maximum value of \( |R_{\text{exact}}| \) over all values of \( \theta \). This choice of normalization keeps errors in the small values of \( R_N \) from dominating \( \text{RE}_{\text{rms}} \). For the tables in this section we will let \( R_{\text{exact}} = R_{120} \) and then demonstrate that the answer has converged to eight digits of accuracy (indicated by \( \text{RE}_{\text{rms}} = 0.0 \) in the tables). Each of the table entries gives \( \text{RE}_{\text{rms}} \) as a function of number of terms \( (N) \) for various quantities calculated. Note that \( \text{RE}_{\text{rms}} \) is the actual error and not a percentage.

For the low frequency, \( f = 2.997925 \times 10^7 \) Hz, the analytical solution has converged to eight digits in 20 terms.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( J_\theta )</th>
<th>( J_\phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>( 0.263 \times 10^{-2} )</td>
<td>( 0.321 \times 10^{-2} )</td>
</tr>
<tr>
<td>10</td>
<td>( 0.280 \times 10^{-1} )</td>
<td>( 0.132 \times 10^{-1} )</td>
</tr>
<tr>
<td>20</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( N )</th>
<th>( E_r )</th>
<th>( E_\theta )</th>
<th>( E_\phi )</th>
<th>( E_\theta^f )</th>
<th>( E_\phi^f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>( 0.359 \times 10^{-2} )</td>
<td>( 0.973 \times 10^{-2} )</td>
<td>( 0.793 \times 10^{-2} )</td>
<td>( 0.175 \times 10^{-5} )</td>
<td>( 0.166 \times 10^{-5} )</td>
</tr>
<tr>
<td>10</td>
<td>( 0.408 \times 10^{-5} )</td>
<td>( 0.290 \times 10^{-5} )</td>
<td>( 0.206 \times 10^{-5} )</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>20</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( N )</th>
<th>( H_r )</th>
<th>( H_\theta )</th>
<th>( H_\phi )</th>
<th>\text{Maximum Error}</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>( 0.121 \times 10^{-1} )</td>
<td>( 0.347 \times 10^{-2} )</td>
<td>( 0.284 \times 10^{-2} )</td>
<td>( 0.121 \times 10^{-1} )</td>
</tr>
<tr>
<td>10</td>
<td>( 0.169 \times 10^{-1} )</td>
<td>( 0.684 \times 10^{-8} )</td>
<td>( 0.584 \times 10^{-8} )</td>
<td>( 0.290 \times 10^{-1} )</td>
</tr>
<tr>
<td>20</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

For the intermediate frequency, \( f = 2.997925 \times 10^8 \) Hz, the analytical solution has converged to eight digits in 30 terms.
Finally, for the high frequency, \( f = 2.997925 \times 10^9 \) Hz, the analytical solution needs 110 terms in order to converge to eight digits.

We conclude that if we use 120 terms in the analytical solution, the solution is accurate to 8 digits and is adequate for comparing to a numerical solution.

### 3.5 Procedures for Checking the Numerical Solution of a PEC Sphere

In this section we outline the procedures that must be followed to compare a numerical solution of the PEC sphere to the analytical solution. We not only have to properly set up the input decks to make the
best comparison possible, but also run additional code to get the numerical data in the proper format for comparison.

To organize the many computer runs needed for comparison, we created separate input files for each test problem and named the input and output files to various codes in the comparison sequence consistently. The `sphr_10cm` part of the file name indicates a sphere gridded with elements having a 10 cm edge length. The number/letter combination part of the file name (0a for example) indicates the equation to be solved and the excitation. The numbers 0, 1 and 2 stand for the EFIE, MFIE and CFIE respectively. The letters a through e indicate different frequencies for the $J_\theta$ solution. The letters f through j indicate different frequencies for the $J_\phi$ solution. An example file name — `sphr_10cm_0a.eig` — indicates that this problem is a PEC sphere gridded with elements having a 10 cm edge length. The sphere is solved with the EFIE, the frequency is 29.979 MHz, and the excitation is a plane wave with $\theta_{inc} = 180^0$, $\phi_{inc} = 315^0$ and $H_\phi = (+2.65442 \times 10^{-3}, 0.0)$ (an excitation suitable to obtain $J_\theta$).

3.5.1 Comparing Surface Currents

1. Grid the surface of a sphere with a regular, triangular mesh. Figure 2 shows the mesh used throughout this report, where the radius of the geometric sphere is 1 meter and the element edges are nominally 10 cm long. **Eiger** performs calculations on the faceted element sphere inscribed within the defined geometric sphere so the numerical solution will be for a sphere with a circumference slightly less than that of the geometric sphere. This difference in size must be taken into account for the most precise comparison with analytical results.

The post-processor, **moench**, can calculate the current density along a line located on the surface of the sphere. The current can be either parallel to, or perpendicular to this line. The line is defined by the user listing a set of element pairs. Each pair defines an edge, which is common to both elements. **Moench** calculates the current density flowing across each edge as well as the cumulative distance from edge to edge.

To calculate $J_\theta$ we choose pairs of elements whose common edges are perpendicular to $\theta$. The centers of the common edges are all located at $\phi_{obs} = 315^0$ and the edges are chosen in an order so that $\theta$ varies from smallest to largest value. The first edge chosen is located an arc distance of 0.0945 meters from the sphere axis. To calculate $J_\phi$ we choose pairs of elements whose common edges are perpendicular to $\phi$ at $\phi_{obs} = 270^0$. Again, the edges are chosen in an order so that $\theta$ varies from smallest to largest value. The
center of the first edge is located an arc distance of 0.0523 meters from the sphere axis. Figure 3 shows a few elements from Figure 2 to illustrate these points. Our strategy throughout this report will be to always keep the grid and these element pairs the same and vary frequency and angle of incidence in order to compare to various analytical quantities.

Since we are comparing to physical currents in the analytical solution, we must make the grid so that the element normals point outward in order to avoid sign errors.

2. Run the pre-processor, jungfrau, to define the electromagnetic problem. The incident plane wave in the analytical solution propagates in the +\( z \) direction. Therefore, we defined the incident plane wave in jungfrau to have \( \theta_{inc} = 180^\circ \). We then choose \( \phi_{inc} \) in jungfrau to orient the incident wave with respect to the elements chosen in Step 1.

For \( J_\theta \) we set \( \phi_{inc} = 315^\circ \) and \( H_{\phi} = (+2.65442 \times 10^{-3}, 0.0) \) A/m, which aligns the incident 1 V/m \( E \) field with the center of the element edges at \( \phi_{obs} = 315^\circ \) and drives the maximum current across these edges. For the analytical solution we will set \( \phi_{obs} = 0^\circ \) to align the observation angle with the incident \( E \) field.

For \( J_\phi \) we set \( \phi_{inc} = 0^\circ \) and \( H_{\phi} = (+2.65442 \times 10^{-3}, 0.0) \) A/m. This drives the maximum \( J_\phi \) across the edges at \( \phi_{obs} = 270^\circ \). For the analytical solution we will also set \( \phi_{obs} = 270^\circ \).

Note that since the edges for \( J_\theta \) and \( J_\phi \) are not at the same value of \( \phi \), (and can never be at the same value of \( \phi \) because of discretization) we must create an input file for each comparison and run eiger multiple times. An example jungfrau input file for \( J_\theta \) is sphr_10cm_0a.in, which is printed in Appendix II.

3. Run eiger. Example input is sphr_10cm_0a.eig. Example output is sphr_10cm_0a.mnh.

4. Run moench asking for unknowns along a line (the ul option) and give the set of element pairs from Step 1 to plot either \( J_\theta \) or \( J_\phi \) along a constant \( \phi \) value. An example moench input file for \( J_\theta \) is moench_sphr10cm_jt.in, which is printed in Appendix II. Output is given in a user-named *.lin1 file (sphr_10cm_0a.lin1, for example).

5. Use a text editor to strip the header information from the *.lin1 file. The zero location of the
*.lin1 file is the element edge designated by the first pair of elements, which is not the zero location of the analytical solution. Run moench_current_offset to add the proper location offset to the first column of the *.lin1 file. For \( J_\theta \) we add 0.0945 meters and for \( J_\phi \) we add 0.0523 meters. An example output file is jt_10cm0a_offset.txt. This completes the numerical portion of the comparison.

6. Run pec_sphere to obtain the analytic solution. The number of terms \( (n) \) is set to be 120 for each run based on what we found in the convergence studies in Section 3.4. The frequency of the analytical solution is identical to that of the numerical solution. The radius of the analytic sphere is slightly smaller than 1.0 meters \((a = 0.9989497 \text{ meters})\) so that the numerical element sphere and the analytic sphere have the same circumference. To compare with the numerical solution of \( J_\theta \), \( \phi_{\text{obs}} \) is set to be 0\(^0\) (recall that the analytic \( E \) field is aligned along \( \phi = 0^0 \)). To compare with the numerical solution of \( J_\phi \), \( \phi_{\text{obs}} \) is set to be 270\(^0\). The observation radius \( r_{\text{obs}} \) is not used in calculating the current and can be set to any value. The input file for \( J_\theta \) (pec_sphere.in) is printed in Appendix II. \( J_\theta \) is output to the file pec_sphere_jt.txt and \( J_\phi \) is output to pec_sphere_jp.txt.

7. Run sphere_surface_compare to find the point-wise relative error and root mean squared relative error (\( R_{\text{rms}} \) given by Equation 3) between pec_sphere_jt.txt and jt_10cm0a_offset.txt. When applying Equation 3 we assume that the analytic solution is the exact solution \( (R_{\text{exact}}) \) and the numerical solution is \( R_N \). Since jt_10cm0a_offset.txt contains non-uniformly spaced data, the code linearly interpolates the analytical solution. Relative error as a function of \( \theta \) is output in error.txt; current as a function of \( \theta \) is output in jt_plot.txt.

### 3.5.2 Comparing Normal E fields

**Eiger** can calculate the \( E \) field normal to and at the centroid of a set of user-designated elements without resorting to moench. The following procedure allows us to compare the numerical \( E_n \) to the analytical solution.

1. Step 1 is the same as in comparing surface currents.

2. Run jungfrau with the same incident field as was used in calculating \( J_\theta \). Request the normal field for the set of elements used for plotting \( J_\theta \). These elements have centroids that are clustered around \( \phi_{\text{obs}} = 315^0 \). The elements are ordered so that \( \theta \) varies from smallest to largest value.

3. Run eiger. Example output file containing the normal fields is sphr_10cm_0a.nor.

4. Run eiger_enormal_offset, which calculates the spherical coordinates of each \( E_n \) element’s centroid. The coordinates will vary non-uniformly in \( \theta \) and may vary in \( \phi \) as well, since due to the construction of the grid, the element centroids may not be perfectly aligned along \( \phi_{\text{obs}} \). **Eiger_enormal_offset** rejects elements whose centroids fall outside a user-defined tolerance (\( \Delta \phi \)) around \( \phi_{\text{obs}} \) and outputs \( E_n \) on the remaining elements to a user-named file. An example output file is enormal_10cm0a_offset.txt. This completes the numerical portion of the comparison.

5. Run pec_sphere with \( \phi_{\text{obs}} = 0.0^0 \) and \( r_{\text{obs}} = 0.9989297 \text{ meters} \) (the sphere radius). Output is pec_sphere_enf.txt.

6. Run sphere_surface_compare. Relative error as a function of \( \theta \) is output in error.txt; \( E_n \) as a function of \( \theta \) is output in jt_plot.txt. Only the first two columns in each file are used.
3.5.3 Comparing Near Fields

In this section we will only consider the \( E \) near field comparison since the \( H \) near field is analogous.

1-3. Steps 1 through 3 are the same as those for comparing the \( J_\theta \) surface current.

4. Run \texttt{sphere\_jfg\_build} to generate a grid of bars along an arc. The bars will define points where near fields are to be calculated. The code generates 179 disconnected bar elements all having centers at user-defined coordinates \( r_{arc} \) and \( \phi_{arc} \). The \( \theta_{arc} \) coordinates range from 0° to 179° in 10° increments. The points and elements are written out in \(*.jfg\) format. This special grid allows us to exactly match the observation points of the analytic case, eliminating the need for interpolation, which was necessary in comparing surface quantities like current and normal \( E \) field. For this test case we generated an arc at \( r_{arc} = 1.1 \) meters and \( \phi_{arc} = 0^0 \). Note that since the input \( E \) field is directed along \( \phi_{inc} = 315^0 \) we are observing the near fields at an angle +45° with respect to the incident \( E \) field. We need to account for this when running the analytic case.

5. Run \texttt{moench} asking for near fields (nf option) on a read-in grid (re option) and give the name of the \(*.jfg\) file generated in Step 4 as the read-in grid. An example input file \texttt{moench\_sphr1.1\_nfl.in} is printed in Appendix II. An example output file is \texttt{sphr10cm0a\_r1.1\_0deg.nfl0}, the name of which indicates that the near field was calculated at a radius of 1.1 meters and \( \phi_{obs} = 0^0 \).

6. Run \texttt{moench\_r2s\_nfl} to convert the total \( E \) and \( H \) fields in \texttt{sphr10cm0a\_r1.1\_0deg.nfl0} from rectangular to spherical coordinates. The output is given in two user-named files: one for \( E \) fields and the other for \( H \) fields. Both files have a format identical to that of the analytic near field file. An example output file for the \( E \) field is named \texttt{enf\_10cm0a.txt}. This completes the numerical portion of the comparison.

7. Run \texttt{pec\_sphere} with \( \phi_{obs} = 45^0 \) and \( r_{obs} = 1.1 \) meters to match the numerical case. An example input file is printed in Appendix II. Output is in \texttt{pec\_sphere\_enf.txt}.

8. Run \texttt{sphere\_field\_compare} to find the relative error as a function of \( \theta \) and RMS error between near field components. Output is in \texttt{error.txt}.

3.5.4 Comparing Far Fields

1-3. Steps 1 through 3 are the same as those for comparing the \( J_\theta \) surface current.

4. Run \texttt{moench} asking for far fields (ff option). Ask for the far fields at a single value of \( \phi = 0^0 \) and 179 values of \( \theta \) from 0° to 179°. An example input file \texttt{moench\_sphr10cm\_ffl.in} is printed in Appendix II. An example output file is named \texttt{sphr10cm0a\_0deg.ffl0}, which indicates that the far field was calculated at \( \phi = 0^0 \).

5. Run \texttt{moench\_ffl} to strip out the scattered far field data. An example output file is named \texttt{sphr10cm0a\_ff.txt}. This completes the numerical portion of the comparison.

6. Run \texttt{pec\_sphere} with \( \phi_{obs} = 45^0 \) to match the numerical case. The quantity \( r_{obs} \) is unimportant since the observation radius is normalized out of the far field calculation. Output is in \texttt{pec\_sphere\_ff.txt}.

7. Run \texttt{sphere\_field\_compare} to find the relative error as a function of \( \theta \) and RMS error between far field components. Output is in \texttt{error.txt}.
3.6 Numerical Results

In this section we compare the numerical solution to analytical solution for all quantities of interest, which are the surface current density, all components of $E$ and $H$ in the near field, and $E_\theta$ and $E_\phi$ in the far field. We make the comparison for each of the equations that can be used by Eiger: the electric field integral equation (EFIE), the magnetic field integral equation (MFIE) and the combined field integral equation (CFIE). For the EFIE calculation we show plots of all the quantities of interest at a single frequency and summarize the RMS relative error for all frequencies in tables. For the other equations we only show results for the surface current density – showing plots for a single frequency – and summarizing the RMS relative error for all frequencies in tables. Results are also shown for all equations when the frequency is such that an internal resonance occurs.

3.6.1 EFIE

In Figures 4 through 14 we show results for the EFIE calculation. In each figure we compare the magnitude of each quantity of interest computed by Eiger (isolated solid triangles) to the quantity calculated analytically (solid lines) as a function of $\theta$. The left hand axis is the magnitude of the quantity. Also plotted in each figure is the relative error (line with open squares), which goes with the right hand axis. The frequency was set at 299.7925 MHz, which corresponds to the rule of thumb that the element edge length should be approximately 1/10 of a wave-length or smaller for good results. The relative error measures both magnitude and phase. Therefore, a high relative error when the magnitude shows good comparison implies that differences in phase are dominating the error. The $\text{RE}_{rms}$ (Equation 3) is evaluated and printed in the subtitle.

In the tables below we study the RMS relative error for all quantities as we vary the frequency, keeping the grid fixed. Column 1 is the frequency in MHz, column 2 is the number of basis functions per wavelength, and the remaining columns shows the RMS relative error for each quantity. The last
Figure 5. Comparison (left axis) and relative error (right axis) of $|J_\phi|$ vs. $\theta$ for $f = 299.7925$ MHz

Figure 6. Comparison (left axis) and relative error (right axis) of $E_n$ vs. $\theta$ for $f = 299.7925$ MHz
Figure 7. Comparison (left axis) and relative error (right axis) of $E_r$ vs. $\theta$ for $f = 299.7925$ MHz

Figure 8. Comparison (left axis) and relative error (right axis) of $E_\theta$ vs. $\theta$ for $f = 299.7925$ MHz
Figure 9. Comparison (left axis) and relative error (right axis) of $E_\phi$ vs. $\theta$ for $f = 299.7925$ MHz

Figure 10. Comparison (left axis) and relative error (right axis) of $E_\theta^{ff}$ vs. $\theta$ for $f = 299.7925$ MHz
Figure 11. Comparison (left axis) and relative error (right axis) of $E_\phi^{ff}$ vs. $\theta$ for $f = 299.7925$ MHz

Figure 12. Comparison (left axis) and relative error (right axis) of $H_r$ vs. $\theta$ for $f = 299.7925$ MHz
Figure 13. Comparison (left axis) and relative error (right axis) of $H_\theta$ vs. $\theta$ for $f = 299.7925$ MHz

Figure 14. Comparison (left axis) and relative error (right axis) of $H_\phi$ vs. $\theta$ for $f = 299.7925$ MHz
The column of the last table shows the maximum error over all the columns. The maximum relative error increases from 0.77% when we discretize at 100 basis functions per wavelength to 37% when we discretize at 5 basis functions per wavelength. The error increases monotonically as the number of basis functions per wavelength decreases. Also note that the maximum error is usually from the \( E_n \) column.

<table>
<thead>
<tr>
<th>Frequency (MHz)</th>
<th>basis/( \lambda )</th>
<th>( J_\theta )</th>
<th>( J_\phi )</th>
<th>( E_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>29.979</td>
<td>100</td>
<td>( 0.77 \times 10^{-3} )</td>
<td>( 0.12 \times 10^{-4} )</td>
<td>( 0.51 \times 10^{-2} )</td>
</tr>
<tr>
<td>199.862</td>
<td>15</td>
<td>( 0.11 \times 10^{-1} )</td>
<td>( 0.77 \times 10^{-2} )</td>
<td>( 0.25 \times 10^{-1} )</td>
</tr>
<tr>
<td>299.793</td>
<td>10</td>
<td>( 0.30 \times 10^{-1} )</td>
<td>( 0.21 \times 10^{-1} )</td>
<td>( 0.56 \times 10^{-1} )</td>
</tr>
<tr>
<td>428.275</td>
<td>7</td>
<td>( 0.83 \times 10^{-1} )</td>
<td>( 0.28 \times 10^{-1} )</td>
<td>0.10</td>
</tr>
<tr>
<td>599.585</td>
<td>5</td>
<td>0.29</td>
<td>0.20</td>
<td>0.37</td>
</tr>
</tbody>
</table>

3.6.2 MFIE

In Figures 15 through 18 we show results for \( |J_\theta| \) and \( |J_\phi| \) solved by all three equations that can be applied to the PEC sphere (EFIE, MFIE, and CFIE) compared to the analytic solution. The discretization is 10 basis functions per wavelength. At this frequency, the CFIE shows the best comparison, followed by the EFIE and finally the MFIE.

The following table shows the RMS error for \( J_\theta \) and \( J_\phi \) at various frequencies when the MFIE is applied.

<table>
<thead>
<tr>
<th>Frequency (MHz)</th>
<th>basis/( \lambda )</th>
<th>( J_\theta )</th>
<th>( J_\phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>29.979</td>
<td>100</td>
<td>( 0.12 \times 10^{-1} )</td>
<td>( 0.13 \times 10^{-1} )</td>
</tr>
<tr>
<td>199.862</td>
<td>15</td>
<td>( 0.24 \times 10^{-1} )</td>
<td>( 0.19 \times 10^{-1} )</td>
</tr>
<tr>
<td>299.793</td>
<td>10</td>
<td>( 0.41 \times 10^{-1} )</td>
<td>( 0.32 \times 10^{-1} )</td>
</tr>
<tr>
<td>428.275</td>
<td>7</td>
<td>( 0.62 \times 10^{-1} )</td>
<td>( 0.32 \times 10^{-1} )</td>
</tr>
<tr>
<td>599.585</td>
<td>5</td>
<td>0.11</td>
<td>0.14</td>
</tr>
</tbody>
</table>

3.6.3 CFIE

The following table shows the RMS error for \( J_\theta \) and \( J_\phi \) at various frequencies when the CFIE is applied. Note that the CFIE has a lower RMS relative error that either the EFIE or MFIE.
Figure 15. Comparison of $|J_\theta|$ vs. $\theta$ as solved by different equations for $f = 299.7925$ MHz

Figure 16. Relative error of $J_\theta$ vs. $\theta$ as solved by different equations for $f = 299.7925$ MHz
Figure 17. Comparison of $|J_\phi|$ vs. $\theta$ as solved by different equations for $f = 299.7925$ MHz

Figure 18. Relative error of $J_\phi$ vs. $\theta$ as solved by different equations for $f = 299.7925$ MHz
Figure 19. Comparison of $|J_\theta|$ vs. $\theta$ at the 289.70 MHz as calculated by the EFIE, MFIE and CFIE. Note that only the CFIE maintains its low RMS relative error, as shown in Figure 20. The error for the other two equations increases significantly.

### 3.6.4 Internal Resonance Test Case

At certain frequencies, even though we are actually solving a scattering problem, the interior of the surface that defines the sphere becomes resonant causing problems for the EFIE and MFIE formulation. These problems manifest themselves in an impedance matrix with a high condition number and a current density corrupted by the sourceless solution of the resonant cavity. One such frequency for the PEC sphere considered here is 289.70 MHz, $(ka = 6.062)$. At this frequency the TM to $\hat{r}$, $n=1$, $m=4$ mode is resonant.

3.6.4 Internal Resonance Test Case

Figures 19 shows $|J_\theta|$ versus $\theta$ at the 289.70 MHz as calculated by the EFIE, MFIE and CFIE. Note that only the CFIE maintains its low RMS relative error, as shown in Figure 20. The error for the other two equations increases significantly.

### 4 Lossless Dielectric Sphere Test Case

#### 4.1 Derivation of the Analytical Solution

If the sphere is a dielectric, with permittivity $\varepsilon_d = \varepsilon_r \varepsilon_0$ and permeability $\mu_d = \mu_r \mu_0$, the potentials
Figure 20. Relative error for $J_\theta$ vs. $\theta$ as solved by different equations at resonance ($f = 289.70$ MHz)
outside the sphere are given as [2]

\[ A_r^+ = \frac{E_0}{\omega \mu_0} \cos \phi \sum_{n=1}^{\infty} \left[ a_n \hat{J}_n(k_0r) + b_n \hat{H}_n^{(2)}(k_0r) \right] P_n^1(\cos \theta) \]

\[ F_r^+ = \frac{E_0}{k_0} \sin \phi \sum_{n=1}^{\infty} \left[ a_n \hat{J}_n(k_0r) + c_n \hat{H}_n^{(2)}(k_0r) \right] P_n^1(\cos \theta) \]

while the potentials inside the sphere are given as

\[ A_r^- = \frac{E_0}{\omega \mu_0} \cos \phi \sum_{n=1}^{\infty} d_n \hat{J}_n(k_d r) P_n^1(\cos \theta) \]

\[ F_r^- = \frac{E_0}{k_0} \sin \phi \sum_{n=1}^{\infty} e_n \hat{J}_n(k_d r) P_n^1(\cos \theta) \]

where \( k_d = \omega \sqrt{\varepsilon_d \mu_d} \) is the wavenumber in the dielectric material,

\[ a_n = \frac{j^{-n}(2n+1)}{n(n+1)} \]

\[ b_n = \frac{\sqrt{\varepsilon_d \mu_0^2} \hat{J}_n(k_0 a) \hat{J}_n(k_d a) + \sqrt{\varepsilon_0 \mu_0^2} \hat{J}_n(k_0 a) \hat{J}_n(k_d a) a_n}{\sqrt{\varepsilon_d \mu_0^2} \hat{H}_n^{(2)}(k_0 a) \hat{J}_n(k_d a) - \sqrt{\varepsilon_0 \mu_0^2} \hat{H}_n^{(2)}(k_0 a) \hat{J}_n(k_d a) a_n} \]

\[ c_n = \frac{\sqrt{\varepsilon_d \mu_0^2} \hat{H}_n^{(2)}(k_0 a) \hat{J}_n(k_d a) + \sqrt{\varepsilon_0 \mu_0^2} \hat{H}_n^{(2)}(k_0 a) \hat{J}_n(k_d a) a_n}{\sqrt{\varepsilon_d \mu_0^2} \hat{H}_n^{(2)}(k_0 a) \hat{J}_n(k_d a) - \sqrt{\varepsilon_0 \mu_0^2} \hat{H}_n^{(2)}(k_0 a) \hat{J}_n(k_d a) a_n} \]

\[ d_n = \frac{\sqrt{\varepsilon_d \mu_0^2} \hat{H}_n^{(2)}(k_0 a) \hat{J}_n(k_d a) - \sqrt{\varepsilon_0 \mu_0^2} \hat{H}_n^{(2)}(k_0 a) \hat{J}_n(k_d a) a_n}{\sqrt{\varepsilon_d \mu_0^2} \hat{H}_n^{(2)}(k_0 a) \hat{J}_n(k_d a) - \sqrt{\varepsilon_0 \mu_0^2} \hat{H}_n^{(2)}(k_0 a) \hat{J}_n(k_d a) a_n} \]

\[ e_n = \frac{\sqrt{\varepsilon_d \mu_0^2} \hat{J}_n^{(2)}(k_0 a) \hat{J}_n(k_d a) - \sqrt{\varepsilon_0 \mu_0^2} \hat{J}_n^{(2)}(k_0 a) \hat{J}_n(k_d a) a_n}{\sqrt{\varepsilon_d \mu_0^2} \hat{J}_n^{(2)}(k_0 a) \hat{J}_n(k_d a) - \sqrt{\varepsilon_0 \mu_0^2} \hat{J}_n^{(2)}(k_0 a) \hat{J}_n(k_d a) a_n} \]

Outside the sphere, the fields are the same as given in Section 3.2 with new values of \( b_n \) and \( c_n \). Inside the sphere the fields are:

\[ E_r = \frac{1}{j \omega \varepsilon_d} \left( \frac{\partial^2}{\partial r^2} + k_d^2 \right) A_r^- \]

\[ = -j E_0 \mu_r \cos \phi \sum_{n=1}^{\infty} d_n \left[ \hat{J}_n^{(2)}(k_d r) + \hat{J}_n(k_d r) \right] P_n^1(\cos \theta) \]

\[ E_\theta = -\frac{1}{r \sin \theta} \frac{\partial F_r^-}{\partial \phi} + \frac{1}{j \omega \varepsilon_d} \frac{\partial^2 A_r^-}{\partial \phi \partial \theta} \]

\[ = -E_0 \cos \phi \sum_{n=1}^{\infty} e_n \hat{J}_n(k_d r) \frac{P_n^1(\cos \theta)}{\sin \theta} - j \sqrt{\frac{\mu_r}{\varepsilon_r}} d_n \hat{J}_n(k_d r) P_n^1(\cos \theta) \sin \theta \]

\[ E_\phi = \frac{1}{r} \frac{\partial F_r^-}{\partial \theta} + \frac{1}{j \omega \varepsilon_d r \sin \theta} \frac{\partial^2 A_r^-}{\partial \phi \partial \theta} \]

\[ = -E_0 \sin \phi \sum_{n=1}^{\infty} e_n \hat{J}_n(k_d r) \frac{P_n^1(\cos \theta)}{\sin \theta} - j \sqrt{\frac{\mu_r}{\varepsilon_r}} d_n \hat{J}_n(k_d r) \frac{P_n^1(\cos \theta)}{\sin \theta} \]

\[ H_r = \frac{1}{j \omega \mu_d} \left( \frac{\partial^2}{\partial r^2} + k_d^2 \right) F_r^- \]

\[ = -j E_0 \frac{\varepsilon_r}{\eta_0} \sin \phi \sum_{n=1}^{\infty} e_n \left[ \hat{J}_n^{(2)}(k_d r) + \hat{J}_n(k_d r) \right] P_n^1(\cos \theta) \]
\[
H_\theta = \frac{1}{r \sin \theta} \frac{\partial A_r^0}{\partial \phi} + \frac{1}{j \omega \mu_r} \frac{\partial^2 F_r^0}{\partial r \partial \theta}
\]
\[
= \frac{-E_0 \sin \phi}{\eta_0 k_0 r} \sum_{n=1}^{\infty} \left[ d_n \tilde{J}_n (k_d r) \frac{P_n^1 (\cos \theta)}{\sin \theta} - j \sqrt{\frac{\varepsilon_r}{\mu_r}} e_n \tilde{J}_n (k_d r) P_n^{1'} (\cos \theta) \sin \theta \right]
\]
\[
H_\phi = \frac{1}{r} \frac{\partial A_r^0}{\partial \theta} + \frac{1}{j \omega \mu_r \sin \theta} \frac{\partial^2 F_r^0}{\partial \theta}
\]
\[
= \frac{E_0 \cos \phi}{\eta_0 k_0 r} \sum_{n=1}^{\infty} \left[ d_n \tilde{J}_n (k_d r) P_n^{1'} (\cos \theta) \sin \theta - j \sqrt{\frac{\varepsilon_r}{\mu_r}} e_n \tilde{J}_n (k_d r) \frac{P_n^1 (\cos \theta) \sin \theta}{\sin \theta} \right]
\]

4.2 Code Implementation of the Analytical Solution

The formulas in Section 4 were implemented in the code \texttt{diel\_sphere}. \texttt{diel\_sphere} calculates quantities of interest versus \( \theta \), which varies from \( 1^\circ \) to \( 179^\circ \) in \( 1^\circ \) increments. The other two observation coordinates, \( \phi_{\text{obs}} \) and \( r_{\text{obs}} \), are set by the user. In the code, \( E_0 = 1.0 \) V/m. Note that the relative permittivity and permeability (\( \varepsilon_r \) and \( \mu_r \)) are both real quantities for the lossless dielectric.

User input to \texttt{diel\_sphere} is interactive and consists of the following:

- Number of terms used in the summations.
- Frequency (Hertz).
- Relative permittivity of dielectric \( \varepsilon_r \) (unitless).
- Relative permeability of dielectric \( \mu_r \) (unitless).
- Sphere radius \( a \) (meters).
- Observation radius \( r_{\text{obs}} \) (meters).
- Observation phi \( \phi_{\text{obs}} \) (degrees).

The results are written to seven output files, each of which contain 179 rows (1 row per \( \theta \) value) and seven columns. When necessary, zeros are put in certain columns of each row in order to make the format consistent. All angles, \( \theta \) and phase, are given in degrees.

Output files:
- \texttt{diel\_sphere\_jt.txt}: \( \theta \), \(|J_\theta|\), phase \( J_\theta \), 0.0,0.0,0.0,0.0,0.0
- \texttt{diel\_sphere\_jp.txt}: \( \theta \), \(|J_\phi|\), phase \( J_\phi \), 0.0,0.0,0.0,0.0,0.0
- \texttt{diel\_sphere\_mt.txt}: \( \theta \), \(|M_\theta|\), phase \( M_\theta \), 0.0,0.0,0.0,0.0,0.0
- \texttt{diel\_sphere\_mp.txt}: \( \theta \), \(|M_\phi|\), phase \( M_\phi \), 0.0,0.0,0.0,0.0,0.0
- \texttt{diel\_sphere\_enf.txt}: \( \theta \), \(|E_r|\), phase \( E_r \), \(|E_\theta|\), phase \( E_\theta \), \(|E_\phi|\), phase \( E_\phi \)
- \texttt{diel\_sphere\_hnf.txt}: \( \theta \), \(|H_r|\), phase \( H_r \), \(|H_\theta|\), phase \( H_\theta \), \(|H_\phi|\), phase \( H_\phi \)
- \texttt{diel\_sphere\_ff.txt}: \( \theta \), 0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0

The first two files (\texttt{diel\_sphere\_jt.txt} and \texttt{diel\_sphere\_jp.txt}) contain the \( \tilde{\theta} \) and \( \tilde{\phi} \).
directed components of electric current density. The next two files (`diel_sphere_mp.txt` and `diel_sphere_mf.txt`) contain the \( \hat{\theta} \) and \( \hat{\phi} \) directed components of magnetic current density. Unlike the PEC case, which specializes the radius to the surface of the sphere to obtain the current, these results are for \( r = r_{\text{obs}} \) (not necessarily specialized to the sphere surface). \( \vec{J} = \hat{r} \times \vec{H} \) and \( \vec{M} = E \times \hat{r} \). Therefore, when comparing current density, \( r_{\text{obs}} \) must be set to be equal to the sphere radius. The next two files (`diel_sphere_enf.txt` and `diel_sphere_hnf.txt`) contain the spherical components of total \( E \) and \( H \) fields in the near field region at \((r_{\text{obs}}, \phi_{\text{obs}})\). The last file (`diel_sphere_ff.txt`) contains the \( \hat{\theta} \) and \( \hat{\phi} \) directed components of \( E^\ast \) in the far field multiplied by the factor \((re^{\ast}k_0r)\).

4.3 Convergence Study of the Analytical Solution

We will examine the convergence behavior of the analytical solution as we increase the number of terms like we did for the PEC sphere. We will again use a sphere radius of \( a = 0.9989497 \) meters. Unlike the PEC sphere, we will observe near field quantities just off the surface of the sphere at \( r_{\text{obs}} = 1.0 \) meters and \( \phi_{\text{obs}} = 45^\circ \) in order to capture the convergence behavior of the surface currents, which are related to the \( \theta \) and \( \phi \) components of field at this location. We will look at convergence for the same frequencies as the PEC case: \( f = 2.997925 \times 10^7 \) Hz, \( f = 2.997925 \times 10^8 \) Hz, and \( f = 2.997925 \times 10^9 \) Hz and we will set \( \varepsilon_r = 3 \), and \( \mu_r = 2 \).

For the following tables we will again use Equation 3, let \( R_{\text{exact}} = R_{120} \) and demonstrate that the answer has converged to eight digits of accuracy (indicated by \( \text{RE}_{\text{rms}} = 0.0 \) in the tables). Each of the table entries gives \( \text{RE}_{\text{rms}} \) as a function of number of terms \( (N) \) for various quantities calculated. Note that \( \text{RE}_{\text{rms}} \) is the actual error and not a percentage.

For the low frequency, \( f = 2.997925 \times 10^7 \) Hz, the analytical solution has converged to eight digits in 20 terms.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( E_r )</th>
<th>( E_\theta )</th>
<th>( E_\phi )</th>
<th>( \text{RE}_{\text{rms}} )</th>
<th>( \text{RE}_{\text{rms}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>( 0.282 \times 10^{-2} )</td>
<td>( 0.248 \times 10^{-2} )</td>
<td>( 0.204 \times 10^{-2} )</td>
<td>( 0.139 \times 10^{-5} )</td>
<td>( 0.128 \times 10^{-5} )</td>
</tr>
<tr>
<td>10</td>
<td>( 0.535 \times 10^{-5} )</td>
<td>( 0.926 \times 10^{-5} )</td>
<td>( 0.220 \times 10^{-8} )</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>20</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( N )</th>
<th>( H_r )</th>
<th>( H_\theta )</th>
<th>( H_\phi )</th>
<th>Maximum Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>( 0.296 \times 10^{-2} )</td>
<td>( 0.256 \times 10^{-2} )</td>
<td>( 0.216 \times 10^{-2} )</td>
<td>( 0.296 \times 10^{-2} )</td>
</tr>
<tr>
<td>10</td>
<td>( 0.472 \times 10^{-5} )</td>
<td>( 0.506 \times 10^{-5} )</td>
<td>( 0.751 \times 10^{-8} )</td>
<td>( 0.926 \times 10^{-8} )</td>
</tr>
<tr>
<td>20</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

For the intermediate frequency, \( f = 2.997925 \times 10^8 \) Hz, the analytical solution has converged to eight digits in 30 terms.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( E_r )</th>
<th>( E_\theta )</th>
<th>( E_\phi )</th>
<th>( \text{RE}_{\text{rms}} )</th>
<th>( \text{RE}_{\text{rms}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>( 0.573 \times 10^{-1} )</td>
<td>0.113</td>
<td>0.132</td>
<td>( 0.109 \times 10^{-2} )</td>
<td>( 0.164 \times 10^{-2} )</td>
</tr>
<tr>
<td>20</td>
<td>( 0.660 \times 10^{-1} )</td>
<td>( 0.178 \times 10^{-1} )</td>
<td>( 0.167 \times 10^{-1} )</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>30</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( N )</th>
<th>( H_r )</th>
<th>( H_\theta )</th>
<th>( H_\phi )</th>
<th>Maximum Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.233</td>
<td>0.142</td>
<td>0.908 ( \times 10^{-1} )</td>
<td>0.233</td>
</tr>
<tr>
<td>20</td>
<td>( 0.319 \times 10^{-1} )</td>
<td>( 0.110 \times 10^{-1} )</td>
<td>( 0.203 \times 10^{-1} )</td>
<td>( 0.660 \times 10^{-1} )</td>
</tr>
<tr>
<td>30</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

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Finally, for the high frequency, \( f = 2.997925 \times 10^9 \) Hz, the analytical solution needs 110 terms in order to converge to eight digits.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( E_r )</th>
<th>( E_\theta )</th>
<th>( E_\phi )</th>
<th>( E_\theta^{1/4} )</th>
<th>( E_\phi^{1/4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.229</td>
<td>0.233</td>
<td>0.231</td>
<td>0.895\times10^{-1}</td>
<td>0.900\times10^{-1}</td>
</tr>
<tr>
<td>50</td>
<td>0.217</td>
<td>0.128</td>
<td>0.150</td>
<td>0.400\times10^{-1}</td>
<td>0.395\times10^{-1}</td>
</tr>
<tr>
<td>70</td>
<td>0.146\times10^{-1}</td>
<td>0.797\times10^{-2}</td>
<td>0.113\times10^{-1}</td>
<td>0.109\times10^{-3}</td>
<td>0.125\times10^{-3}</td>
</tr>
<tr>
<td>90</td>
<td>0.126\times10^{-1}</td>
<td>0.667\times10^{-8}</td>
<td>0.101\times10^{-1}</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>100</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>110</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

### 4.4 Procedures for Checking the Numerical Solution of a Dielectric Sphere

In this section we outline the procedures that must be followed to compare a numerical solution of the dielectric sphere to the analytical solution. All of the files are tagged with the name **sphr_10cm** and a number/letter combination. The number part (3) is for the dielectric equation and the letter part (a through t) is for various frequencies and excitations.

#### 4.4.1 Comparing Surface Currents

1. We use the same grid as the PEC case, but since the wavelength inside the dielectric is \( 1/\sqrt{3} \) the wavelength outside the dielectric, we will lower the frequency in order to obtain the required number of basis functions per wavelength inside the dielectric. We will use the same element pairs to define edges as in the PEC case.

2. Run **jungfrau**. To compare \( J_\theta \) and \( J_\phi \) we use the same incident wave as we did in the PEC test case. In addition to electric current density, the dielectric sphere has magnetic current density \( M \). For all surface current calculations we set \( \theta_{inc} = 180^0 \).

   For \( M_\theta \) we set \( \phi_{inc} = 45^0 \) and \( H_\phi = \left( +2.65442 \times 10^{-3}, 0.0 \right) \) A/m, which aligns the incident 1 V/m E field 90\(^0\) with respect to the center of the element edges at \( \phi_{obs} = 315^0 \) and drives the maximum \( M_\theta \) across these edges. For the analytical solution we will set \( \phi_{obs} = 270^0 \) to align the observation angle 90\(^0\) with respect to the incident E field.

   For \( M_\phi \) we set \( \phi_{inc} = 270^0 \) and \( H_\theta = \left( +2.65442 \times 10^{-3}, 0.0 \right) \) A/m. This drives the maximum \( M_\phi \) across the edges at \( \phi_{obs} = 270^0 \). For the analytical solution we set \( \phi_{obs} = 0^0 \).

3. Run **eiger**. Example input is **sphr_10cm_3a.eig**. Example output is **sphr_10cm_3a.mnh**.

4. Run **moench** asking for unknowns along a line (the ul option) and give the set of element pairs from Step 1 of the PEC case to plot \( J_\theta \), \( J_\phi \), \( M_\theta \), or \( M_\phi \) along a constant \( \phi \) value. An example **moench** input file for \( M_\phi \) is **moench_sphr10cm_mp.in**, which is shown in Appendix II. Output is given in a
5. Use a text editor to strip the header information from the *.lin1 file. The zero location of the *.lin1 file is the first element edge designated in the element pairs, which is not the zero location of the analytical solution. Run moench_current_offset to add the proper location offset to the first column of the *.lin1 file. For $J_\theta$ or $M_\theta$ we add 0.0945 meters and for $J_\phi$ or $M_\phi$ we add 0.0523 meters. An example output file is mp_10cm3c_offset.txt. This completes the numerical portion of the comparison.

6. Run diel_sphere to obtain the analytic solution. The number of terms ($n$) is set to be 120 based on the convergence studies in Section 4.3. The frequency of the analytical solution is identical to that of the numerical solution. The radius of the analytic sphere is $a = 0.9989497$ meters. To compare with the numerical solution of $J_\theta$, $\phi_{obs}$ is set to be $0^\circ$. To compare with the numerical solution of $J_\phi$, $\phi_{obs}$ is set to be $270^\circ$. To compare with the numerical solution of $M_\theta$, $\phi_{obs}$ is set to be $270^\circ$. To compare with the numerical solution of $M_\phi$, $\phi_{obs}$ is set to be $0^\circ$. The observation radius $r_{obs}$ must be set to be the same as the sphere radius, so $r_{obs} = 0.9989297$ meters. An example input file to calculate $M_\phi$ (diel_sphere.in) is printed in Appendix II. $J_\theta$ is output to the file diel_sphere_jt.txt, $J_\phi$ is output to diel_sphere jp.txt, $M_\theta$ to diel_sphere mt.txt, and $M_\phi$ to diel_sphere mp.txt.

7. Run sphere_surface_compare to find the point-wise relative error and root mean squared relative error ($RE_{rms}$ given by Equation 3) between the analytical and numerical results.

4.4.2 Comparing Near Fields

Near fields are compared exactly like the PEC case except the analytical solution is found by running diel_sphere.

4.4.3 Comparing Far Fields

Far fields are compared exactly like the PEC case except the analytical solution is found by running diel_sphere.

4.5 Numerical Results

In Figures 25 through 32, we follow the procedures of Section 4.4 to compare the numerical results to the analytical results. In each figure we compare the magnitude of each quantity of interest computed by Eiger (isolated solid triangles) to the quantity calculated analytically (solid lines) as a function of $\theta$. The left hand axis is the magnitude of the quantity. Also plotted in each figure is the relative error (line with open squares), which goes with the right hand axis. All of these results are at $f = 122.38978$ MHz where the elements have an edge length of $1/10$ of the wavelength inside the dielectric sphere.

In the tables below we study the RMS relative error for all quantities as we vary the frequency, keeping the grid fixed. The frequency in column 1 along with the nominal size of the element (10cm) yields the number of basis functions per wavelength shown in column 2. The wavelength is that inside the dielectric sphere, $(1/\sqrt{6}$ of the wavelength outside) so the frequency is uniformly lower than the PEC test cases. The remaining columns show the RMS relative error for each quantity of interest.

The table below shows the $RE_{rms}$ for all components of surface current density.
Figure 21. Comparison (left axis) and relative error (right axis) of $E_r$ outside the dielectric sphere vs. $\theta$ for $f = 122.38978$ MHz.

Figure 22. Comparison (left axis) and relative error (right axis) of $E_r$ inside the dielectric sphere vs. $\theta$ for $f = 122.38978$ MHz.
Figure 23. Comparison (left axis) and relative error (right axis) of $E_\theta$ outside the dielectric sphere vs. $\theta$ for $f = 122.38978$ MHz

Figure 24. Comparison (left axis) and relative error (right axis) of $E_\theta$ inside the dielectric sphere vs. $\theta$ for $f = 122.38978$ MHz
Figure 25. Comparison (left axis) and relative error (right axis) of $E_\phi$ outside the dielectric sphere vs. $\theta$ for $f = 122.38978$ MHz

Figure 26. Comparison (left axis) and relative error (right axis) of $E_\phi$ inside the dielectric sphere vs. $\theta$ for $f = 122.38978$ MHz
Figure 27. Comparison (left axis) and relative error (right axis) of $H_r$ outside the dielectric sphere vs. $\theta$ for $f = 122.38978$ MHz

Figure 28. Comparison (left axis) and relative error (right axis) of $H_r$ inside the dielectric sphere vs. $\theta$ for $f = 122.38978$ MHz
Figure 29. Comparison (left axis) and relative error (right axis) of $H_\theta$ outside the dielectric sphere vs. $\theta$ for $f = 122.38978$ MHz

Figure 30. Comparison (left axis) and relative error (right axis) of $H_\theta$ inside the dielectric sphere vs. $\theta$ for $f = 122.38978$ MHz
Figure 31. Comparison (left axis) and relative error (right axis) of $H_{\phi}$ outside the dielectric sphere vs. $\theta$ for $f = 122.38978$ MHz

Figure 32. Comparison (left axis) and relative error (right axis) of $H_{\phi}$ inside the dielectric sphere vs. $\theta$ for $f = 122.38978$ MHz
When \( r_{\text{obs}} = 1.1 \) meters (outside the sphere) the RMS errors for the \( E \) field components are shown in columns 3 through 5 in the following table. Columns 6 and 7 are the far field errors.

<table>
<thead>
<tr>
<th>Frequency (MHz)</th>
<th>basis/( \lambda )</th>
<th>( J_0 )</th>
<th>( J_\phi )</th>
<th>( M_0 )</th>
<th>( M_\phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.239</td>
<td>100</td>
<td>0.48\times10^{-1}</td>
<td>0.78\times10^{-1}</td>
<td>0.55\times10^{-1}</td>
<td>0.90\times10^{-1}</td>
</tr>
<tr>
<td>81.593</td>
<td>15</td>
<td>0.13\times10^{-1}</td>
<td>0.14\times10^{-1}</td>
<td>0.10\times10^{-1}</td>
<td>0.91\times10^{-2}</td>
</tr>
<tr>
<td>122.390</td>
<td>10</td>
<td>0.11\times10^{-1}</td>
<td>0.11\times10^{-1}</td>
<td>0.95\times10^{-2}</td>
<td>0.11\times10^{-1}</td>
</tr>
<tr>
<td>174.843</td>
<td>7</td>
<td>0.65\times10^{-2}</td>
<td>0.74\times10^{-2}</td>
<td>0.53\times10^{-2}</td>
<td>0.68\times10^{-2}</td>
</tr>
<tr>
<td>244.780</td>
<td>5</td>
<td>0.11\times10^{-1}</td>
<td>0.10\times10^{-1}</td>
<td>0.13\times10^{-1}</td>
<td>0.11\times10^{-1}</td>
</tr>
</tbody>
</table>

The next table shows RMS errors for the \( H \) near field at \( r_{\text{obs}} = 1.1 \) m.

<table>
<thead>
<tr>
<th>Frequency (MHz)</th>
<th>basis/( \lambda )</th>
<th>( H_r )</th>
<th>( H_\theta )</th>
<th>( H_\phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.239</td>
<td>100</td>
<td>0.60\times10^{-2}</td>
<td>0.83\times10^{-2}</td>
<td>0.10\times10^{-1}</td>
</tr>
<tr>
<td>81.593</td>
<td>15</td>
<td>0.51\times10^{-2}</td>
<td>0.54\times10^{-2}</td>
<td>0.67\times10^{-2}</td>
</tr>
<tr>
<td>122.390</td>
<td>10</td>
<td>0.13\times10^{-1}</td>
<td>0.52\times10^{-2}</td>
<td>0.42\times10^{-2}</td>
</tr>
<tr>
<td>174.843</td>
<td>7</td>
<td>0.71\times10^{-2}</td>
<td>0.43\times10^{-2}</td>
<td>0.39\times10^{-2}</td>
</tr>
<tr>
<td>244.780</td>
<td>5</td>
<td>0.24\times10^{-1}</td>
<td>0.11\times10^{-1}</td>
<td>0.15\times10^{-1}</td>
</tr>
</tbody>
</table>

When \( r_{\text{obs}} = 0.9 \) meters (inside the sphere), the following tables show the errors in the \( E \) and \( H \) field. The final column is the maximum error found for all other columns.

<table>
<thead>
<tr>
<th>Frequency (MHz)</th>
<th>basis/( \lambda )</th>
<th>( E_r )</th>
<th>( E_\theta )</th>
<th>( E_\phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.239</td>
<td>100</td>
<td>0.21\times10^{-1}</td>
<td>0.17\times10^{-1}</td>
<td>0.20\times10^{-1}</td>
</tr>
<tr>
<td>81.593</td>
<td>15</td>
<td>0.15\times10^{-1}</td>
<td>0.14\times10^{-1}</td>
<td>0.14\times10^{-1}</td>
</tr>
<tr>
<td>122.390</td>
<td>10</td>
<td>0.67\times10^{-2}</td>
<td>0.63\times10^{-2}</td>
<td>0.62\times10^{-2}</td>
</tr>
<tr>
<td>174.843</td>
<td>7</td>
<td>0.11\times10^{-1}</td>
<td>0.72\times10^{-2}</td>
<td>0.66\times10^{-2}</td>
</tr>
<tr>
<td>244.780</td>
<td>5</td>
<td>0.18\times10^{-1}</td>
<td>0.13\times10^{-1}</td>
<td>0.12\times10^{-1}</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Frequency (MHz)</th>
<th>basis/( \lambda )</th>
<th>( H_r )</th>
<th>( H_\theta )</th>
<th>( H_\phi )</th>
<th>Max Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.239</td>
<td>100</td>
<td>0.19\times10^{-1}</td>
<td>0.15\times10^{-1}</td>
<td>0.19\times10^{-1}</td>
<td>0.90\times10^{-1}</td>
</tr>
<tr>
<td>81.593</td>
<td>15</td>
<td>0.16\times10^{-1}</td>
<td>0.12\times10^{-1}</td>
<td>0.13\times10^{-1}</td>
<td>0.16\times10^{-1}</td>
</tr>
<tr>
<td>122.390</td>
<td>10</td>
<td>0.10\times10^{-1}</td>
<td>0.59\times10^{-2}</td>
<td>0.52\times10^{-2}</td>
<td>0.13\times10^{-1}</td>
</tr>
<tr>
<td>174.843</td>
<td>7</td>
<td>0.14\times10^{-1}</td>
<td>0.56\times10^{-2}</td>
<td>0.57\times10^{-2}</td>
<td>0.14\times10^{-1}</td>
</tr>
<tr>
<td>244.780</td>
<td>5</td>
<td>0.23\times10^{-1}</td>
<td>0.11\times10^{-1}</td>
<td>0.90\times10^{-2}</td>
<td>0.30\times10^{-1}</td>
</tr>
</tbody>
</table>

Surprisingly, although the errors are small, they do not increase monotonically with frequency as they did in the PEC case. Rather, at 100 basis functions per wavelength the error is less than 9.0%, then it decreases until at 10 basis functions per wavelength the error is less than 1.3% and then it increases again until at 5 basis functions per wavelength it is less than 3%.
5 Lossy Dielectric Sphere Test Case

The equations for the lossy dielectric case are the same as those in Section 4 except that \( \mu_r \) and \( \varepsilon_r \) are both complex, with a positive real part and negative imaginary part. The quantities \( k_d \) and the Bessel functions are also complex.

5.1 Code Implementation of the Analytical Solution

The formulas in Section 4 were implemented in the code lossy_sphere. Lossy_sphere calculates quantities of interest versus \( \theta \), which varies from \( 1^0 \) to \( 179^0 \) in \( 1^0 \) increments. The other two observation coordinates, \( \phi_{\text{obs}} \) and \( r_{\text{obs}} \), are set by the user. In the code, \( E_\theta = 1.0 \, \text{V/m} \).

User input to lossy_sphere consists of the following:

- Number of terms used in the summations.
- Frequency (Hertz).
- Relative permittivity of dielectric (unitless).
- Relative permeability of dielectric (unitless).
- Sphere radius \( a \) (meters).
- Observation radius \( r_{\text{obs}} \) (meters).
- Observation phi \( \phi_{\text{obs}} \) (degrees).

The results are written to three output files, each of which contain 179 rows (1 row per \( \theta \) value) and seven columns.

Output files:
- lossy_sphere_it.txt: \( \theta, |J_\theta|, \text{phase } J_\theta, 0.0, 0.0, 0.0, 0.0 \)
- lossy_sphere_jp.txt: \( \theta, |J_\phi|, \text{phase } J_\phi, 0.0, 0.0, 0.0, 0.0 \)
- lossy_sphere_mt.txt: \( \theta, |M_\theta|, \text{phase } M_\theta, 0.0, 0.0, 0.0, 0.0 \)
- lossy_sphere_mp.txt: \( \theta, |M_\phi|, \text{phase } M_\phi, 0.0, 0.0, 0.0, 0.0 \)
- lossy_sphere_enf.txt: \( \theta, |E_r|, \text{phase } E_r, |E_\theta|, \text{phase } E_\theta, |E_\phi|, \text{phase } E_\phi \)
- lossy_sphere_hnf.txt: \( \theta, |H_r|, \text{phase } H_r, |H_\theta|, \text{phase } H_\theta, |H_\phi|, \text{phase } H_\phi \)
- lossy_sphere_ff.txt: \( \theta, 0.0, 0.0, |r e^{jkr} E_{\theta}^{\text{ff}}|, \text{phase } r e^{jkr} E_{\theta}^{\text{ff}}, |r e^{jkr} E_{\phi}^{\text{ff}}|, \text{phase } r e^{jkr} E_{\phi}^{\text{ff}} \)

These files are analogous to the output files of diel_sphere.

5.2 Convergence Study of the Analytical Solution

We will examine the convergence behavior of the analytical solution as we increase the number of terms. We will use the same parameters as the lossless dielectric sphere except that we will set \( \varepsilon_r = (3.0, -0.2) \), and \( \mu_r = (2.0, -0.1) \).

For the following tables we will let \( R_{\text{exact}} = R_{200} \) and then demonstrate that the answer has converged
to eight digits of accuracy (indicated by $\text{RE}_{\text{rms}} = 0.0$ in the tables). Each of the table entries gives $\text{RE}_{\text{rms}}$ (defined in Equation 3) as a function of number of terms ($N$) for various quantities calculated.

For the low frequency, $f = 2.997925 \times 10^7$ Hz, the analytical solution has converged to eight digits in 20 terms.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$E_r$</th>
<th>$E_\theta$</th>
<th>$E_\phi$</th>
<th>$E_{\theta}^{ij}$</th>
<th>$E_{\phi}^{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.283$\times 10^{-2}$</td>
<td>0.240$\times 10^{-2}$</td>
<td>0.205$\times 10^{-2}$</td>
<td>0.140$\times 10^{-3}$</td>
<td>0.129$\times 10^{-3}$</td>
</tr>
<tr>
<td>10</td>
<td>0.586$\times 10^{-8}$</td>
<td>0.790$\times 10^{-8}$</td>
<td>0.372$\times 10^{-8}$</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>20</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

For the intermediate frequency, $f = 2.997925 \times 10^8$ Hz, the analytical solution has converged to eight digits in 30 terms.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$E_r$</th>
<th>$E_\theta$</th>
<th>$E_\phi$</th>
<th>$E_{\theta}^{ij}$</th>
<th>$E_{\phi}^{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.297$\times 10^{-2}$</td>
<td>0.249$\times 10^{-2}$</td>
<td>0.211$\times 10^{-2}$</td>
<td>0.297$\times 10^{-2}$</td>
<td>0.297$\times 10^{-2}$</td>
</tr>
<tr>
<td>10</td>
<td>0.548$\times 10^{-8}$</td>
<td>0.467$\times 10^{-8}$</td>
<td>0.665$\times 10^{-8}$</td>
<td>0.790$\times 10^{-8}$</td>
<td>0.790$\times 10^{-8}$</td>
</tr>
<tr>
<td>20</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

For the high frequency, $f = 2.997925 \times 10^9$ Hz, the analytical solution needs 120 terms in order to converge to eight digits.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$E_r$</th>
<th>$E_\theta$</th>
<th>$E_\phi$</th>
<th>$E_{\theta}^{ij}$</th>
<th>$E_{\phi}^{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.420</td>
<td>0.279</td>
<td>0.278</td>
<td>0.391$\times 10^{-1}$</td>
<td>0.394$\times 10^{-1}$</td>
</tr>
<tr>
<td>70</td>
<td>0.441$\times 10^{-1}$</td>
<td>0.222$\times 10^{-1}$</td>
<td>0.220$\times 10^{-1}$</td>
<td>0.911$\times 10^{-3}$</td>
<td>0.890$\times 10^{-4}$</td>
</tr>
<tr>
<td>90</td>
<td>0.597$\times 10^{-1}$</td>
<td>0.337$\times 10^{-1}$</td>
<td>0.322$\times 10^{-1}$</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>120</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Finally, for the high frequency, $f = 2.997925 \times 10^9$ Hz, the analytical solution needs 120 terms in order to converge to eight digits.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$H_r$</th>
<th>$H_\theta$</th>
<th>$H_\phi$</th>
<th>Maximum Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.420</td>
<td>0.279</td>
<td>0.278</td>
<td>0.391$\times 10^{-1}$</td>
</tr>
<tr>
<td>70</td>
<td>0.441$\times 10^{-1}$</td>
<td>0.222$\times 10^{-1}$</td>
<td>0.220$\times 10^{-1}$</td>
<td>0.911$\times 10^{-3}$</td>
</tr>
<tr>
<td>90</td>
<td>0.597$\times 10^{-1}$</td>
<td>0.337$\times 10^{-1}$</td>
<td>0.322$\times 10^{-1}$</td>
<td>0.0</td>
</tr>
<tr>
<td>120</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

5.3 Numerical Results

We follow the same procedures as in Section 4.4 to compare the numerical results to the analytical results.
When $r_{obs} = 1.1$ meters (outside the sphere)

<table>
<thead>
<tr>
<th>Frequency (MHz)</th>
<th>basis/λ</th>
<th>$J_{θ}$</th>
<th>$J_{φ}$</th>
<th>$M_{θ}$</th>
<th>$M_{φ}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.239</td>
<td>100</td>
<td>0.48×10^{-1}</td>
<td>0.77×10^{-1}</td>
<td>0.54×10^{-1}</td>
<td>0.89×10^{-1}</td>
</tr>
<tr>
<td>81.593</td>
<td>15</td>
<td>0.17×10^{-1}</td>
<td>0.16×10^{-1}</td>
<td>0.12×10^{-1}</td>
<td>0.11×10^{-1}</td>
</tr>
<tr>
<td>122.390</td>
<td>10</td>
<td>0.11×10^{-1}</td>
<td>0.12×10^{-1}</td>
<td>0.13×10^{-1}</td>
<td>0.11×10^{-1}</td>
</tr>
<tr>
<td>174.843</td>
<td>7</td>
<td>0.12×10^{-1}</td>
<td>0.12×10^{-1}</td>
<td>0.11×10^{-1}</td>
<td>0.11×10^{-1}</td>
</tr>
<tr>
<td>244.780</td>
<td>5</td>
<td>0.13×10^{-1}</td>
<td>0.12×10^{-1}</td>
<td>0.14×10^{-1}</td>
<td>0.12×10^{-1}</td>
</tr>
</tbody>
</table>

When $r_{obs} = 0.9$ meters (inside the sphere)

<table>
<thead>
<tr>
<th>Frequency (MHz)</th>
<th>basis/λ</th>
<th>$E_{r}$</th>
<th>$E_{θ}$</th>
<th>$E_{φ}$</th>
<th>$E_{r'}$</th>
<th>$E_{θ'}$</th>
<th>$E_{φ'}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.239</td>
<td>100</td>
<td>0.60×10^{-2}</td>
<td>0.82×10^{-2}</td>
<td>0.10×10^{-1}</td>
<td>0.11×10^{-2}</td>
<td>0.15×10^{-2}</td>
<td></td>
</tr>
<tr>
<td>81.593</td>
<td>15</td>
<td>0.38×10^{-2}</td>
<td>0.41×10^{-2}</td>
<td>0.47×10^{-2}</td>
<td>0.23×10^{-2}</td>
<td>0.24×10^{-2}</td>
<td></td>
</tr>
<tr>
<td>122.390</td>
<td>10</td>
<td>0.78×10^{-2}</td>
<td>0.47×10^{-2}</td>
<td>0.52×10^{-2}</td>
<td>0.21×10^{-2}</td>
<td>0.18×10^{-2}</td>
<td></td>
</tr>
<tr>
<td>174.843</td>
<td>7</td>
<td>0.63×10^{-2}</td>
<td>0.38×10^{-2}</td>
<td>0.44×10^{-2}</td>
<td>0.12×10^{-2}</td>
<td>0.14×10^{-2}</td>
<td></td>
</tr>
<tr>
<td>244.780</td>
<td>5</td>
<td>0.10×10^{-1}</td>
<td>0.55×10^{-2}</td>
<td>0.72×10^{-2}</td>
<td>0.18×10^{-2}</td>
<td>0.30×10^{-2}</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Frequency (MHz)</th>
<th>basis/λ</th>
<th>$H_{r}$</th>
<th>$H_{θ}$</th>
<th>$H_{φ}$</th>
<th>Max Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.239</td>
<td>100</td>
<td>0.21×10^{-1}</td>
<td>0.16×10^{-1}</td>
<td>0.20×10^{-1}</td>
<td>Max Error</td>
</tr>
<tr>
<td>81.593</td>
<td>15</td>
<td>0.13×10^{-1}</td>
<td>0.12×10^{-1}</td>
<td>0.12×10^{-1}</td>
<td>0.12×10^{-1}</td>
</tr>
<tr>
<td>122.390</td>
<td>10</td>
<td>0.96×10^{-2}</td>
<td>0.83×10^{-2}</td>
<td>0.82×10^{-2}</td>
<td>0.82×10^{-2}</td>
</tr>
<tr>
<td>174.843</td>
<td>7</td>
<td>0.19×10^{-1}</td>
<td>0.97×10^{-2}</td>
<td>0.97×10^{-2}</td>
<td>0.97×10^{-2}</td>
</tr>
<tr>
<td>244.780</td>
<td>5</td>
<td>0.17×10^{-1}</td>
<td>0.13×10^{-1}</td>
<td>0.13×10^{-1}</td>
<td>0.13×10^{-1}</td>
</tr>
</tbody>
</table>

In terms of convergence as a function of frequency the lossy dielectric sphere behaves much like the lossless dielectric sphere.

### 6 Conclusions

In this report we demonstrated that Eiger can calculate the current density, near fields and far fields for plane wave scattering from three-dimensional objects made of either PEC, lossless dielectric, or lossy dielectric. All available equations were tested. The ability of the CFIE to overcome internal resonance problems was also demonstrated. If the gridding is sufficiently fine (better than 10 basis functions per wavelength) the relative error should be less than the 10% error required by Test 1.a.a. in the EMR.
Verification and Validation Plan [1].

7 References


8 Appendix I: Special Functions

8.1 Associated Legendre Polynomials $P^1_n(x)$

For the calculations discussed in this report, the order of the associated Legendre polynomial is always 1 and degree is an integer, $n = 1...N$. The results will be stored in the array $P1(n)$ where $n$ represents the degree. The first two values in the sequence are calculated explicitly.

$$P1(1) = P^1_1(x) = -(1 - x^2)^{1/2}$$

$$P1(2) = P^1_2 = -3x(1 - x^2)^{1/2}$$

The remaining values, $n = 3...N$, are calculated using a recursion relation [3].

$$(n + 1 - m)R^m_{n+1}(x) - (2n + 1)xR^m_n(x) + (n + m)R^m_{n-1}(x) = 0$$

Let $m = 1$

$$R^1_{n+1}(x) = \frac{(2n+1)}{n}xR^1_n(x) - \frac{(n+1)}{n}R^1_{n-1}(x)$$

$$P1(n+1) = \frac{2n+1}{n}xP1(n) - \frac{n+1}{n}P1(n-1)$$

We checked the result using tables found in [3].

8.2 First Derivative of Associated Legendre Polynomials $P^1_n'(x)$

Again the order is always 1 and degree is an integer 1 to $N$. The results will be stored in an array $DP1(n)$ where $n$ represents the degree.
Let \( m = 1 \), therefore,

\[
\frac{dP^m}{dx} = \frac{1}{1-x^2} \left[ -nxP^m(x) + (n + m) P^m_{n-1}(x) \right]
\]

\[
\frac{dP^1}{dx} = \frac{1}{1-x^2} \left[ -nP^1(x) + (1 + 1) P^1_{n-1}(x) \right]
\]

\[
DP^1(n) = \frac{1}{1-x^2} [-nP^1(n) + (1 + 1) P^1(n - 1)]
\]

For the special case of \( n = 1 \)

\[
\frac{dP^1}{dx} = \frac{x}{(1-x^2)^{1/2}}
\]

\[
DP^1(1) = \frac{x}{(1-x^2)^{1/2}}
\]

We checked the above result against a finite-difference approximation of the derivative using \( P^1_n(x) \).

### 8.3 Alternative Spherical Bessel Functions \( \hat{B}_n \)

The alternative spherical Bessel function of order \( n \) is

\[
\hat{B}_n(x) = \sqrt{\frac{\pi x}{2}} B_{n+1/2}(x)
\]

where \( B_n \) represents the Bessel function of the first or second kind, and \( n = 1 \) to \( N \). \( B_{n+1/2}(x) \) is calculated using the SLATEC [4] routines BESJ or BESY and stored in \( B1(n) \). We checked the result using tables found in [3].

### 8.4 First Derivative of Alternative Spherical Bessel Function \( \hat{B}'_n \)

\[
\frac{d\hat{B}_n(x)}{dx} = -\sqrt{\frac{\pi x}{2}} B_{n+1+1/2}(x) + \left( \sqrt{\frac{\pi}{2x}} (n+1) \right) B_{n+1/2}(x)
\]

\[
DB1(n) = -\sqrt{\frac{\pi x}{2}} B1(n + 1) + \left( \sqrt{\frac{\pi}{2x}} (n+1) \right) B1(n)
\]

We checked the derivative against a finite difference approximation of \( \hat{B}'_n \) using \( \hat{B}_n \).

### 8.5 Second Derivative of Alternative Spherical Bessel Function \( \hat{B}''_n \)

\[
\frac{d^2\hat{B}_n(x)}{dx^2} = -\sqrt{\frac{\pi x}{2}} B_{n+2+1/2}(x) - \sqrt{\frac{\pi}{2x}} (2n+3) B_{n+1+1/2}(x) + \sqrt{\frac{\pi}{2x^3}} (n+1) n B_{n+1/2}(x)
\]

\[
DB2(n) = -\sqrt{\frac{\pi x}{2}} B1(n + 2) - \sqrt{\frac{\pi}{2x}} (2n+3) B1(n + 1) + \sqrt{\frac{\pi}{2x^3}} (n+1) n B1(n)
\]
We checked the second derivative against a finite difference approximation of $\hat{B}_n^r$ using $\hat{B}_n^r$.

9 Appendix II - Example Input Files

9.1 Electric Current Density

The following input to *jungfrau (sphr_10cm_0a.in)* is used to obtain an *.eig file for the PEC sphere in order to calculate $J_\theta$, $E_n$, all components of $E$ and $H$ in the near field and $E$ in the far field.

```
continue
continue
ce ! create an *.eig file option
sphere_10cm ! *.jfg file (input)
spfr_10cm_0a ! *.eig file (output)
ya ! overwrite all
1 meter radius, 10cm edge, PEC sphere, theta=180, phi=315, f=2.997925e7
3 dimensional problem
dynamic problem
no ground planes
no periodicity
meter
body1 ! green property
sm ! solid metal
pec ! material of sphere
efie ! equation
linear ! interpolation order
away ! element normals point away from metal
1 ! region surrounding metal
no symmetry
continue
homogeneous ! region 1 definition
(1.0,0.0) ! relative permittivity
(1.0,0.0) ! relative permeability
continue
no movement
0 lumped loads
ot ! excitations defined one at a time
1 ! one excitation defined
plane wave ! the excitation is a plane wave
ot ! the plane waves are defined one at a time
1 ! 1 plane wave is defined
1 ! plane wave is in region 1
180.0 ! theta value
315.0 ! phi value
(0.0,0.0) ! H theta
(+2.65442e-3,0.0) ! H phi
ot ! frequencies one at a time
1 ! one frequency
2.997925e7 ! frequency
```
bl ! blanket solutions
lu ! LU decomposition
0 ! 0 far field patterns
58 ! E normal defined at 58 elements
2168 ! Element 1
2146 ! Element 2
2145
2116
2115
2086
2085
2056
2055
2026
2025
1996
1995
1966
1965
1936
1935
1906
1905
1876
1875
1846
1845
1816
1815
1786
1785
1756
1755
15
16
45
46
75
76
105
106
135
136
165
166
195
196
225
226
255
256
285
The following input to `moench (moench_sphr10cm_jt.in)` writes the unknowns associated with $J_\theta$ along a line defined by the user.

```
sphr_10cm_0a *.eig file (input)
sphr_10cm_0a *.mnh file (input)
ul unknowns along a line option
sphr_10cm_0a *.lin1 file (output)
ya overwrite all
1 frequency id
1 excitation id
exit exit information loop
1 1 node_set_id 1, unknown_id 1 (electric current)
29 number of element pairs
2168 pair 1
2146
2145 pair 2
2116
2115 pair 3 etc...
2086
2085
2056
2055
2026
2025
1996
1995
1966
1965
1936
1935
1906
1905
1876
1875
1846
1845
1816
1815
1786
1785
1756

! Element 58
```
The following input to **pec_sphere** is to compare $J_\theta$. Note that the $\phi_{\text{obs}} = 0.0$.

<table>
<thead>
<tr>
<th>number of terms</th>
<th>frequency</th>
<th>sphere radius</th>
<th>observation radius</th>
<th>observation phi</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>29.97925e+06</td>
<td>0.9989497</td>
<td>0.9989497</td>
<td>0.0</td>
</tr>
</tbody>
</table>

**9.2 Near Fields**

The following input to **moench** (**moench_sphr1.1_nfld.in**) is to calculate the near field along an arc defined by the *.jfg file sphere_1p1_0d*. Note that the *.eig file used here must be the same as the *.eig file used to generate the *.mnh file*. If the *.eig file used here is for a different excitation or frequency, the near fields will be incorrect.

<table>
<thead>
<tr>
<th>.eig file</th>
<th>.mnh file</th>
<th>neared field option</th>
<th>read in grid</th>
</tr>
</thead>
<tbody>
<tr>
<td>sphr_10cm_3a</td>
<td>sphr_10cm_3a</td>
<td>nf</td>
<td>re</td>
</tr>
</tbody>
</table>
1
region
sphere_1p1_0d *.jfg grid for near fields
yes output a near field for this excitation
sphr_10cm3a_r1.1_0deg name of near field file
ya overwrite existing file
quit quit moench

The following input to pec_sphere is to compare near fields. Note that the $\phi_{obs} = 45^0$ and $r_{obs} = 1.1$ meters

120 number of terms
29.97925e+06 frequency
0.9989497 sphere radius
1.1 observation radius
45.0 observation phi

9.3 Far Fields

The following input to moench (moench_sphr10cm_fld.in) is to calculate the far field along an arc defined by the user. Note that the *.eig file used here must be the same as the *.eig file used to generate the *.mnh file. If the *.eig file used here is for a different excitation or frequency, the far fields will be incorrect.

sphr_10cm_3a *.eig
sphr_10cm_3a *.mnh
ff far field option
1 number of far_field patterns
1 region id
179 number angles in theta
1.0 first theta angle
179.0 last theta angle
1 number of angles in phi
0.0 phi angle
yes make a plot for excitation and frequency
sphr10cm3e_ff_0deg far field file
ya overwrite the old files
quit quit moench

9.4 Magnetic Current Density

The following input to moench (moench_sphr10cm_mp.in) writes the unknowns associated with $M_{\phi}$ along a line defined by the user.

sphr_10cm_3d *.eig file (input)
sphr_10cm_3d *.mnh file (input)
ul unknowns along a line option
sphr_10cm_3d *.lin1 file (output)
ya overwrite all
1 frequency id
1 excitation id

56
<table>
<thead>
<tr>
<th>exit</th>
<th>exit the information loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>node_set_id 1, unknown_id 2 (magnetic current)</td>
</tr>
<tr>
<td>2</td>
<td>number of element pairs</td>
</tr>
<tr>
<td>30</td>
<td>pair 1</td>
</tr>
<tr>
<td>2596</td>
<td>pair 2</td>
</tr>
<tr>
<td>2175</td>
<td>pair 3 etc...</td>
</tr>
<tr>
<td>2566</td>
<td></td>
</tr>
<tr>
<td>2160</td>
<td></td>
</tr>
<tr>
<td>2536</td>
<td></td>
</tr>
<tr>
<td>2130</td>
<td></td>
</tr>
<tr>
<td>2506</td>
<td></td>
</tr>
<tr>
<td>2100</td>
<td></td>
</tr>
<tr>
<td>2476</td>
<td></td>
</tr>
<tr>
<td>2070</td>
<td></td>
</tr>
<tr>
<td>2446</td>
<td></td>
</tr>
<tr>
<td>2040</td>
<td></td>
</tr>
<tr>
<td>2416</td>
<td></td>
</tr>
<tr>
<td>2010</td>
<td></td>
</tr>
<tr>
<td>2386</td>
<td></td>
</tr>
<tr>
<td>1980</td>
<td></td>
</tr>
<tr>
<td>2356</td>
<td></td>
</tr>
<tr>
<td>1950</td>
<td></td>
</tr>
<tr>
<td>2326</td>
<td></td>
</tr>
<tr>
<td>1920</td>
<td></td>
</tr>
<tr>
<td>2296</td>
<td></td>
</tr>
<tr>
<td>1890</td>
<td></td>
</tr>
<tr>
<td>2266</td>
<td></td>
</tr>
<tr>
<td>1860</td>
<td></td>
</tr>
<tr>
<td>2236</td>
<td></td>
</tr>
<tr>
<td>1830</td>
<td></td>
</tr>
<tr>
<td>2206</td>
<td></td>
</tr>
<tr>
<td>1800</td>
<td></td>
</tr>
<tr>
<td>2176</td>
<td></td>
</tr>
<tr>
<td>1770</td>
<td></td>
</tr>
<tr>
<td>465</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>495</td>
<td></td>
</tr>
<tr>
<td>31</td>
<td></td>
</tr>
<tr>
<td>525</td>
<td></td>
</tr>
<tr>
<td>61</td>
<td></td>
</tr>
<tr>
<td>555</td>
<td></td>
</tr>
<tr>
<td>91</td>
<td></td>
</tr>
<tr>
<td>585</td>
<td></td>
</tr>
<tr>
<td>121</td>
<td></td>
</tr>
<tr>
<td>615</td>
<td></td>
</tr>
<tr>
<td>151</td>
<td></td>
</tr>
<tr>
<td>645</td>
<td></td>
</tr>
<tr>
<td>181</td>
<td></td>
</tr>
<tr>
<td>675</td>
<td></td>
</tr>
<tr>
<td>211</td>
<td></td>
</tr>
<tr>
<td>705</td>
<td></td>
</tr>
<tr>
<td>241</td>
<td></td>
</tr>
<tr>
<td>735</td>
<td></td>
</tr>
</tbody>
</table>
The following input to `diel_sphere` is to compare $M_\phi$. Note that the $\phi_{obs} = 0.0$ and $r_{obs}$ equals the radius of the sphere.

```
50          number of terms
12.23898E+06  frequency
3.0          epsilon
2.0          mu
0.9989497     sphere radius
0.9989497     observation radius
0.0          observation phi
```

**Distribution:**

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