WORKSHOP:
Geometrical interpretation of Radial and Oblique Return Methods

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Tensors are vectors!

To a mathematician, a vector is a member of a set for which *addition* and *scalar multiplication* satisfy certain rules.

Most familiar 3D vector concepts and theorems also apply to tensors when regarded as 9D vectors.

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<tr>
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<th>9D tensor operations</th>
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**3D inner product**

\[ \vec{r} \cdot \vec{s} \text{ means } \sum_{i=1}^{3} r_i s_i \]

**9D inner product**

\[ R : S \text{ means } \sum_{i=1}^{3} \sum_{j=1}^{3} R_{ij} S_{ij} \]
Projection operations

Orthogonal projection

\[ p = x - n(n \cdot x) \]

Oblique projection

\[ p = x - \frac{a(b \cdot x)}{a \cdot b} \]

Note: \( \overset{\sim}{b} \) defines the target plane; \( a \) defines projection direction.

Analog for 9D tensor space:

\[ P(X) = X - \frac{A(B \cdot X)}{A \cdot B} \]

Projections are linear. . . \( P(\alpha_1 X_1 + \alpha_2 X_2) = \alpha_1 P(X_1) + \alpha_2 P(X_2) \)
If there is a $\beta$ such that
\[ x = y + \beta a, \] then
\[ P(x) = P(y). \]

Important: *converse is true too!*

**Analog for tensors:**

If \[ X = Y + \beta A \] then \[ P(X) = P(Y) \] and vice versa.

**Corollary:** \[ P(P(X)) = P(X) \] (projecting twice makes no change).
Nonhardening plasticity

Known:

$B$, gradient of yield function ($B_{ij} = \frac{\partial f}{\partial \sigma_{ij}}$).

$\dot{\varepsilon}$, total strain rate.

$E$, fourth-order elastic tangent stiffness tensor.

$M$, direction of the plastic strain rate.

Unknown:

$\dot{\sigma}$, rate of stress

$\varepsilon^e$, elastic part of the strain rate

$\varepsilon^p$, plastic part of the strain rate.

$\lambda$, magnitude of the plastic part of the strain rate.
Governing equations

\[ \dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p \]

strain rate decomposition

\[ \dot{\varepsilon}^p = \lambda M \]

plastic strain direction is known

\[ \sigma = E : \dot{\varepsilon}^e \]

stress linear in elastic strain

\[ B : \dot{\sigma} = 0 \]

stress stays on yield surface

Solution:

Note \( \dot{\varepsilon}^e = \dot{\varepsilon} - \dot{\varepsilon}^p = \dot{\varepsilon} - \lambda M \) so \( \dot{\sigma} = E : (\dot{\varepsilon} - \lambda M) \). For convenience,

define \( \dot{\sigma}^{\text{trial}} = E : \dot{\varepsilon} \) and \( \dot{A} = E : M \). Then \( \dot{\sigma} = \dot{\sigma}^{\text{trial}} - \lambda \dot{A} \)

Enforce last equation to get \( B : [\dot{\sigma}^{\text{trial}} - \lambda \dot{A}] = 0 \). Solve for \( \lambda \) and back substitute to get solution for stress rate: \( \dot{\sigma} = \dot{\sigma}^{\text{trial}} - \left( \frac{B : \dot{\sigma}^{\text{trial}}}{B : \dot{A}} \right) \dot{A} \).
Slightly rearrange solution to final form:

\[
\dot{\sigma} = P(\dot{\sigma}^{\text{trial}}) \quad \text{where} \quad P(X) = X - \frac{A(B \cdot X)}{A : B}
\]

Numerical solution: \( \sigma = \sigma^{\text{trial}} + \beta A \). Find \( \beta \) by \( f(\sigma^{\text{trial}} + \beta A) = 0 \).
The return direction is...

- coaxial with $\hat{\mathbf{A}}$.
- not generally normal to the yield surface.
- not generally aligned with the plastic strain rate.
- not dictated by physical considerations such as positive dissipation, yield surface convexity, or plastic stability. (Such concerns dictate appropriate values for “known” quantities.)
- “radial” if and only if the material is plastically incompressible.

The above analysis can be generalized (see web document) to include hardening/softening. Projection of the trial stress back to the current yield surface remains valid even though the stress rate is no longer a projection of the trial stress rate.
Equivalent plastic strain

Many constitutive models use yield surface evolution laws that depend on the so-called “equivalent plastic strain,” which is defined

$$\gamma_p \equiv \int \sqrt{\frac{2}{3}} \dot{\varepsilon}' : \dot{\varepsilon}' dt = \sqrt{\frac{2}{3}} \int \dot{\varepsilon}' : \dot{\varepsilon}' dt$$

The best method uses the definition directly:

$$\Delta \gamma_p \equiv \sqrt{\frac{2}{3}} \left\| \dot{\varepsilon}' - \dot{\varepsilon}^e \right\| \Delta t$$, or, for isotropic, $$\Delta \gamma_p \equiv \sqrt{\frac{2}{3}} \left\| \dot{\varepsilon}' - \frac{S_{\text{new}} - S_{\text{old}}}{2G} \right\| \Delta t$$.

For a finite time step $$\Delta t$$,

$$\gamma_p^{\text{new}} \equiv \gamma_p^{\text{old}} + \sqrt{\frac{2}{3}} \left\| \dot{\varepsilon}' \Delta t - \frac{S_{\text{new}} - S_{\text{old}}}{2G} \right\|$$

(...better suited for partially plastic intervals.)
Supplemental topic: 9D vector basis

Recall that tensors are 9D vectors, so we may define a $9 \times 1$ component array for them: $T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9 = \{ T_{11}, T_{21}, T_{31}, T_{12}, T_{22}, T_{32}, T_{13}, T_{23}, T_{33} \}$

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3D inner product

$$r \cdot s \text{ means } \sum_{k=1}^{3} r_k s_k$$

9D inner product

$$R: S \text{ means } \sum_{K=1}^{9} R_K S_K$$

However, yield functions are defined for stress, which is symmetric.
Subspace of symmetric tensors

If a tensor $T$ is symmetric, then $T_{ij} = T_{ji}$, so only 6 of the 9 components are independent.

Voigt: $\{T\}^V = \{T_{11}, T_{22}, T_{33}, T_{23}, T_{31}, T_{12}\}$
then $R:S$ means $R_1^y S_1^y + R_2^y S_2^y + R_3^y S_3^y + 2(R_4^y S_4^y + R_5^y S_5^y + R_6^y S_6^y)$

Mandel: $\{T\}^M = \{T_{11}, T_{22}, T_{33}, \sqrt{2} T_{23}, \sqrt{2} T_{31}, \sqrt{2} T_{12}\}$
Then $R:S$ means $R_1^y S_1^y + R_2^y S_2^y + R_3^y S_3^y + R_4^y S_4^y + R_5^y S_5^y + R_6^y S_6^y$

Q: Is the Mandel convention just a "trick" likely to bite us some day?
A: NO! Voigt components are the dangerous choice — they are referenced to an irregular basis for symmetric tensors. Mandel components are referenced to the same basis, except that it is normalized!
Mandel basis for symmetric tensors

The Mandel basis for 9D full tensor space is

\[ \xi_1 = \varepsilon_1 \varepsilon_1, \quad \xi_2 = \varepsilon_2 \varepsilon_2, \quad \xi_3 = \varepsilon_3 \varepsilon_3, \]
\[ \xi_4 = \frac{1}{\sqrt{2}} (\varepsilon_2 \varepsilon_3 + \varepsilon_3 \varepsilon_2), \quad \xi_5 = \frac{1}{\sqrt{2}} (\varepsilon_3 \varepsilon_1 + \varepsilon_1 \varepsilon_3), \quad \xi_6 = \frac{1}{\sqrt{2}} (\varepsilon_1 \varepsilon_2 + \varepsilon_2 \varepsilon_1) \]
\[ \xi_7 = \frac{1}{\sqrt{2}} (\varepsilon_3 \varepsilon_2 - \varepsilon_2 \varepsilon_3), \quad \xi_8 = \frac{1}{\sqrt{2}} (\varepsilon_1 \varepsilon_3 - \varepsilon_3 \varepsilon_1), \quad \xi_9 = \frac{1}{\sqrt{2}} (\varepsilon_2 \varepsilon_1 - \varepsilon_1 \varepsilon_2) \]

The basis is orthogonal because \( \xi_K : \xi_J = 0 \) if \( K \neq J \). The basis is normalized (i.e., \( \xi_K : \xi_J = \delta_{KJ} \)) because of the factors of \( \sqrt{2} \).

Just as an ordinary vector has components \( v_k = \mathbf{v} \cdot \varepsilon_k \), the Mandel components of a tensor \( T \) are \( T_K = T : \xi_K \). 

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The Rendulic plane plots “shear stress” versus “mean stress.”

**Common choice**

“shear stress:” \( \tau = \sqrt{\frac{1}{2} \mathbf{S} : \mathbf{S}} \), and

“mean stress:” \( p = \frac{1}{3} \text{tr} \mathbf{\sigma} \). Then \( \mathbf{\sigma} = \mathbf{S} + p \mathbf{I} \).

**Problem:** This \( \tau \) vs. \( p \) space isn’t isomorphic to stress space. For example, \( \mathbf{\sigma} : \mathbf{\sigma} \neq \tau^2 + p^2 \). Importantly, the normal to the yield surface in \( \tau \) vs. \( p \) space is *not* normal to the yield surface in stress space.

**Isomorphic choice**

“shear stress” measure: \( \sigma_s = \sqrt{\mathbf{S} : \mathbf{S}} = \mathbf{\sigma} : \mathbf{\hat{S}} \)

“mean stress” measure: \( \sigma_m = \frac{1}{\sqrt{3}} \text{tr} \mathbf{\sigma} = \mathbf{\sigma} : \mathbf{\hat{I}} \)
Supplemental Topic:
Anisotropic yield surfaces

For elastically anisotropic material, a very common “first-cut” best guess at the plastic yield surface is a Tsai-Wu ellipsoid of the form

\[ f(\sigma) = (\sigma - \sigma^*) : L : (\sigma - \sigma^*) - 1, \]

(contrary to Walker’s recent claims, this form is capable of modelling even highly anisotropic media)

where \( L \) shares the same anisotropy with the stiffness \( E \).

Elastic constants may be nondestructively measured, but the yield \( L_{ijkl} \) parameters are more difficult since a fresh sample must be used to measure each component. Thus, data are often lacking.

Proposal: Face with a dearth of data, assume that \( E \) and \( L \) have the same eigenprojectors...
What are eigenprojectors?

To illustrate, consider simpler 3D space. Here’s a sample tensor

\[
[A] = \begin{bmatrix}
17 & -2 & -2 \\
-2 & 14 & -4 \\
-2 & -4 & 14
\end{bmatrix}
\]

which has eigenpairs

\[
\lambda_1 = 9 \quad v_1 = \frac{1}{3}\{1, 2, 2\} \\
\lambda_2 = 18 \quad v_2 = \frac{1}{\sqrt{5}}\{-2, 0, 1\} \\
\lambda_3 = 18 \quad v_3 = \frac{1}{3\sqrt{5}}\{-2, 5, -4\}
\]

In spectral form,

\[
A = \lambda_1 v_1 v_1 + \lambda_2 v_2 v_2 + \lambda_3 v_3 v_3
\]

\[
= 9v_1 v_1 + 18(v_2 v_2 + v_3 v_3)
\]

\[
P_{\approx 1} \quad \quad P_{\approx 2}
\]

unique!

With respect to the principal basis,

\[
A = \begin{bmatrix}
9 & 0 & 0 \\
0 & 18 & 0 \\
0 & 0 & 18
\end{bmatrix}
\]

\[
P_{\approx 1} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

and

\[
P_{\approx 2} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
What are eigentensors?

We seek tensors $Y$ and scalars $\lambda$ such that $E \cdot Y = \lambda Y$. The major and minor symmetries allow this to be written as an ordinary $6 \times 6$ matrix eigenproblem:

$$
\begin{bmatrix}
    E_{1111} & E_{1122} & E_{1133} & \sqrt{2}E_{1211} & \sqrt{2}E_{1222} & \sqrt{2}E_{1233} \\
    E_{2211} & E_{2222} & E_{2233} & \sqrt{2}E_{2311} & \sqrt{2}E_{2322} & \sqrt{2}E_{2333} \\
    E_{3311} & E_{3322} & E_{3333} & \sqrt{2}E_{3111} & \sqrt{2}E_{3122} & \sqrt{2}E_{3133} \\
    \sqrt{2}E_{2311} & \sqrt{2}E_{2322} & \sqrt{2}E_{2333} & 2E_{2312} & 2E_{2321} & 2E_{2332} \\
    \sqrt{2}E_{3111} & \sqrt{2}E_{3122} & \sqrt{2}E_{3133} & 2E_{3112} & 2E_{3121} & 2E_{3132} \\
    \sqrt{2}E_{1211} & \sqrt{2}E_{1222} & \sqrt{2}E_{1233} & 2E_{1212} & 2E_{1221} & 2E_{1231} \\
\end{bmatrix}
\begin{bmatrix}
    Y_{11} \\
    Y_{22} \\
    Y_{33} \\
    \sqrt{2}Y_{23} \\
    \sqrt{2}Y_{31} \\
    \sqrt{2}Y_{32} \\
\end{bmatrix}
= \lambda
\begin{bmatrix}
    Y_{11} \\
    Y_{22} \\
    Y_{33} \\
    \sqrt{2}Y_{23} \\
    \sqrt{2}Y_{31} \\
    \sqrt{2}Y_{32} \\
\end{bmatrix}
$$

An eigensolver will give six orthonormal 6-dimensional eigenvectors. Each of these correspond to symmetric eigentensors.
If \( \lambda \) has multiplicity of 1, then \( P_{ijkl} = Y_{ij}Y_{kl} \) is the corresponding eigenprojector. When it operates on any tensor, the result is the part of that tensor in the direction of \( Y_{ij} \).

EXAMPLE: For isotropy, \( 3K \) is an eigenvalue of multiplicity 1. The eigentensor is \( I/\sqrt{3} \). The projector is \( \frac{1}{3} \delta_{ij} \delta_{kl} \), which merely returns the isotropic part of any tensor it operates on.

If \( \lambda \) has multiplicity of 2, then the eigentensors \( Y^{(1)} \) and \( Y^{(2)} \) are not unique. Instead, the eigenprojector, \( P_{ijkl} = Y^{(1)}_{ij}Y^{(1)}_{kl} + Y^{(2)}_{ij}Y^{(2)}_{kl} \) is unique. When it operates on an arbitrary tensor, the result is the part of the tensor in the subspace. Higher multiplicities are similar.

EXAMPLE: For isotropy, \( 2G \) is an eigenvalue of multiplicity 5. The eigenprojector returns the deviator of any tensor it operates on. Thus, ANY DEVIATORIC TENSOR is an eigentensor for isotropy.
Back to anisotropic yield...

Recall $f(\bar{\sigma}) = (\bar{\sigma} - \sigma^*) L: (\bar{\sigma} - \sigma^*) - 1$. If the material is transverse, the Mandel stiffness is of the form

$$
\begin{bmatrix}
E_0 & E_2 & E_3 & 0 & 0 & 0 \\
E_2 & E_0 & E_3 & 0 & 0 & 0 \\
E_3 & E_3 & E_1 & 0 & 0 & 0 \\
0 & 0 & 0 & E_4 & 0 & 0 \\
0 & 0 & 0 & 0 & E_5 & 0 \\
0 & 0 & 0 & 0 & 0 & E_5
\end{bmatrix}
\begin{bmatrix}
Y_{11} \\
Y_{22} \\
Y_{33} \\
\sqrt{2}Y_{23} \\
\sqrt{2}Y_{31} \\
\sqrt{2}Y_{23}
\end{bmatrix} = \lambda
\begin{bmatrix}
Y_{11} \\
Y_{22} \\
Y_{33} \\
\sqrt{2}Y_{23} \\
\sqrt{2}Y_{31} \\
\sqrt{2}Y_{23}
\end{bmatrix},
$$

where

$$
E_1 = E_{3333}, \ E_2 = E_{1122}, \ E_3 = E_{1133}, \ E_4 = 2E_{2323}, \ E_5 = 2E_{1212}, \ E_0 = E_2 + E_5.
$$

There are five independent stiffnesses, but only four independent eigenvalues (and therefore only four independent eigenprojectors). Forcing $L$ to have the same eigenprojectors gives a formula for the elusive $L_{1133}$ value that couples lateral and axial response.
Conclusions

This presentation covered many applications that illustrate the usefulness of regarding tensors as higher-dimensional vectors.

Key points were

- For radial and oblique return models, the stress may be returned to the yield surface via a projection operation that is analogous to projecting a simple vector onto a plane.
- Symmetric tensors are analogous to planes. The Mandel convention for symmetric tensor components correspond to an orthonormal basis for symmetric tensors.
- The isomorphic stress measures are a more accurate representation of stress space that is analogous to viewing the stress “vector” in the “plane” formed by the isotropic tensor and the stress itself.
- Anisotropic yield may be coupled to elastic isotropy via the elastic eigenprojectors.